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18.969 Topics in Geometry: Mirror Symmetry
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MIRROR SYMMETRY: LECTURE 23

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Recall that given an (almost) Calabi-Yau manifold (X, J, ω, Ω) , we defined M to be the set of pairs (L, ∇) , $L \subset X$ a special Lagrangian torus, ∇ a flat $U(1)$ connection on $\mathbb{C} \times L$ modulo gauge equivalence. Up to 2π ,

$$(1) \quad \begin{aligned} T_{(L, \nabla)}M &= \{(v, i\alpha) \in C^\infty(NL) \oplus \Omega^1(L; i\mathbb{R}) \mid -\iota_v \omega + \frac{1}{2\pi} i\alpha \in \mathcal{H}_\psi^1(L; \mathbb{C})\} \\ &= H_\psi^1(L, \mathbb{C}) = \{\beta \in \Omega^1(L, \mathbb{R}) \mid d\beta = 0, d^*(\psi\beta) = 0\}, \psi = |\Omega|_g \end{aligned}$$

is a complex vector space, giving us an integrable J^\vee on M with holomorphic local coordinates

$$(2) \quad z_\beta(L, \nabla) = \underbrace{\exp(-2\pi\omega(\beta))}_{\mathbb{R}_+} \underbrace{\text{hol}_\nabla(\gamma)}_{U(1)} \in \mathbb{C}^*$$

and a holomorphic $(n, 0)$ -form

$$(3) \quad \Omega^\vee((v_1, i\alpha_1), \dots, (v_n, i\alpha_n)) = i^{-n} \int_L (-\iota_{v_1} \omega + \frac{i\alpha_1}{2\pi}) \wedge \dots \wedge (-\iota_{v_n} \omega + \frac{i\alpha_n}{2\pi})$$

After normalizing $\int_L \Omega = 1$, we obtained a compatible Kähler form

$$(4) \quad \omega^\vee((v_1, i\alpha_1), (v_2, i\alpha_2)) = \frac{1}{2\pi} \int_L \alpha_2 \wedge (\iota_{v_1} \text{Im } \Omega) - \alpha_1 \wedge (\iota_{v_2} \text{Im } \Omega)$$

Now, let B be the set of special Lagrangian tori, $\pi^\vee : M \rightarrow B$, $(L, \nabla) \rightarrow L$ a special Lagrangian torus fibration (with torus fiber $\{(0, i\alpha)\}$) “dual to $\pi : X \rightarrow B$ ”. Note that π^\vee has a zero section $\{(L, d)\}$ which is a special Lagrangian, and has complex conjugation $(L, \nabla) \leftrightarrow (L, \nabla^*)$.

Example. As usual, let $T^2 = \mathbb{C}/\mathbb{Z} + i\rho\mathbb{Z}$, $\Omega = dz$, $\omega = \frac{\lambda}{\rho} dx \wedge dy$, $\int_{T^2} \omega = \lambda$. L is special Lagrangian $\Leftrightarrow \text{Im } dz|_L = 0 \Leftrightarrow L$ is parallel to the real axis. We have a fibration $T^2 \xrightarrow{\pi} S^1 = \mathbb{R}/\rho\mathbb{Z}$, $(x, y) \mapsto y$, with fibers $L_t = \{y = t\}$, inducing a complex affine structure with affine coordinate y ($= \text{Im } \Omega$ on the arc from L_0 to L), $\text{size}(S^1) = \rho$, and a symplectic affine structure $\frac{\lambda}{\rho} y$ ($=$ the symplectic area swept), $\text{size}(S^1) = \lambda$. On the mirror $M = \{(L, \nabla)\} \in \mathbb{R}/\rho\mathbb{Z}$, the holomorphic coordinate for J^\vee is $\exp(-2\pi\frac{\lambda}{\rho}y)e^{i\theta}$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, $\nabla = d + i\theta dx$. Or, taking $\frac{1}{2\pi i} \log$, $z^\vee = \frac{\theta}{2\pi} + i\frac{\lambda}{\rho}y \in \mathbb{C}/\mathbb{Z} + i\lambda\mathbb{Z}$. Furthermore $\Omega^\vee = dz^\vee$, $\omega^\vee = \frac{1}{2\pi} d\theta \wedge dy$. Our SYZ transformation exchanges Lagrangian sections of π and flat connections with a connection on a holomorphic line bundle. Explicitly, set $L = \{x = f(y)\}$, $f :$

$\mathbb{R}/\rho\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$, with flat connection $\nabla = d + ih(y)dy, h : \mathbb{R}/\rho\mathbb{Z} \rightarrow \mathbb{R}$. We build a Hermitian connection $\check{\nabla} = d + if(y)d\theta + ih(y)dy$ on a locally trivialized Hermitian line bundle \mathcal{L} . Note that changing the trivialization by $e^{i\theta}$ changes the connection form by $id\theta$, i.e. $f \leftrightarrow f + 1$, glue $y = 0$ to $y = \rho$ by $e^{i \deg(f)\theta}$. Furthermore, $\deg(\mathcal{L}) = \deg(f : S^1 \rightarrow S^1)$. We have a holomorphic structure $\bar{\partial}^{\check{\nabla}} = \check{\nabla}^{0,1}$.

In higher-dimensional tori, we have $L = \{x = f(y)\}$ Lagrangian, $f : \mathbb{R}^n/\Lambda \rightarrow \mathbb{R}^n/\mathbb{Z}^n$, $\nabla = d + i \sum_j h_j(y)dy_j, h : \mathbb{R}^n/\Lambda \rightarrow \mathbb{R}^n$ on the one side, $\check{\nabla} = d + i \sum_j f_j(y)d\theta_j + i \sum_j h_j(y)dy_j$, which is holomorphic \Leftrightarrow the curvature is $(1, 1)/J^\vee$. Set

$$(5) \quad F = i \sum_{j,k} \frac{\partial f_j}{\partial y_k} dy_k \wedge d\theta_j + i \sum_{j,k} \frac{\partial h_j}{\partial y_k} dy_k \wedge dy_j$$

Then J^\vee exchanges dy_k and $d\theta_k$ up to canonical scaling, and is holomorphic \Leftrightarrow

- $\frac{\partial f_j}{\partial y_k} = \frac{\partial f_k}{\partial y_j}$ for $\sum f_j dy_j$ a closed 1-form on \mathbb{R}^n/Λ ($\Leftrightarrow L$ Lagrangian),
- $\frac{\partial h_j}{\partial y_k} = \frac{\partial h_k}{\partial y_j}$ for $\sum h_j dy_j$ a closed 1-form ($\Leftrightarrow \nabla$ is flat).

Example. Let X be a K3 surface, namely a simply connected complex surface with $K_X \cong \mathcal{O}_X$, e.g. a 4-dimensional hypersurface $\{P_4(x_0, \dots, x_3) = 0\} \subset \mathbb{C}\mathbb{P}^3$ for P_4 a homogeneous polynomial in degree 4, or a double cover of $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$, $\{z^2 = P_{4,4}((x_0, x_1), (y_0, y_1))\} \subset \text{Tot}(\mathcal{O}(2, 2))$ with Hodge diamond

$$(6) \quad \begin{array}{ccc} 1 & 0 & 1 \\ 0 & 20 & 0 \\ 1 & 0 & 1 \end{array}$$

Any K3 surface is *hyperkähler*, i.e. there are three complex structures $I, J, K = IJ = -JI$ inducing three Kähler forms $\omega_I, \omega_J, \omega_K$ for the same Kähler metric g . The idea is the following: given $I, [\omega_I]$, Yau's theorem gives a Ricci-flat Kähler metric g , and we obtain a holomorphic volume form $\Omega_I \in \Omega^{2,0}$ with $|\Omega_I|_g = 1, \Omega_U = \omega_J + i\omega_K$, where ω_I is $(1, 1)$ for $I, \omega_J = \text{Re } \Omega_I, \omega_K = \text{Im } \Omega_I$ $(2, 0) + (0, 2)$ for I are pointwise orthonormal self-dual 2-forms which are covariantly constant.

Some (not all) K3 surfaces admit fibrations by elliptic curves over spheres, typically with 24 nodal singular fibers. For instance, given a double coordinate of $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$, we project to a $\mathbb{C}\mathbb{P}^1$ factor, and observe that the fibers are double covers of $\mathbb{C}\mathbb{P}^1$ branched at four points. Now, assume we have one of these with a holomorphic section. The fibers will be I -complex curves, and thus special Lagrangian for $(\omega_J, \Omega_J = \omega_K + i\omega_I), (\omega_K, \Omega_K = \omega_I + i\omega_J)$ (they are calibrated by ω_I , which is $(1, 1)$ for I so ω_J, ω_K vanish). Mirror symmetry corresponds these latter two structures.