

# Chapter 11

## Introduction to Minimal Surface II

### 11.1 Why a Minimal Surface is Minimal (or Critical)

We want to show why a regular surface  $im(x) = S$  with mean curvature  $H = 0$  everywhere is called a minimal surface – i.e., that this is the surface of least area among all surfaces with the same boundary  $\gamma$  (and conversely, that a surface that minimizes area (for a given boundary  $\gamma$ ) has  $H = 0$  everywhere.) To do this we first use *normal variations* to derive a formula for the change in area in terms of mean curvature, and then as an application of our formula we find that a surface has minimal area if and only if it has mean curvature 0.

Let  $D$  be the domain on which  $x$  is defined, and let  $\gamma$  be a closed curve in  $D$  which bounds a subdomain  $\Delta$ . (This is the notation used in Osserman, p. 20 - 23.) We choose a differentiable function  $N(u)$  (here  $u = (u_1, u_2)$  is a point in our domain  $D$ ) normal to  $S$  at  $u$ , i.e.,

$$N(u) \cdot \frac{\delta x}{\delta u_i} = 0. \quad (11.1)$$

Differentiating yields

$$\frac{\delta N}{\delta u_j} \cdot \frac{\delta x}{\delta u_i} = -N \cdot \frac{\delta^2}{\delta u_i \delta u_j} = -b_{ij}(N). \quad (11.2)$$

Now let  $h(u)$  be an arbitrary  $C^2$  function in  $D$ , and for every real number  $\lambda$  let the surface  $S_\lambda$  be given by

$$S_\lambda : y(u) = x(u) + \lambda h(u)N(u) \quad (11.3)$$

$y$  is called a **normal variation** of  $x$  – since we are varying the surface  $x$  via the parameter  $\lambda$  along the direction of our normal  $N$ . Letting  $A(\lambda)$  denote the area of the surface  $S_\lambda$ , we will show that:

**Theorem 11.1.1.**  $A'(0) = -2 \int \int_S H(N)h(u)dA$ ,

where the integral of  $f$  with respect to surface area on  $S$  is defined as

$$\int \int_S f(u)dA = \int \int_\Delta f(u) \sqrt{\det g_{ij}} du_1 du_2 \quad (11.4)$$

( $A'(0)$  denotes the derivative with respect to  $\lambda$ .)

*Proof.* Differentiating  $y$  with respect to the domain coordinates  $u_i$ , we get

$$\frac{\delta y}{\delta u_i} = \frac{\delta x}{\delta u_i} + \lambda \left( h \frac{\delta N}{\delta u_i} + \frac{\delta h}{\delta u_i} N \right) \quad (11.5)$$

If we let  $g_{ij}^\lambda$  denote the entries of the first fundamental form for the surface  $S_\lambda$ , we get

$$g_{ij}^\lambda = \frac{\delta y}{\delta u_i} \cdot \frac{\delta y}{\delta u_j} = g_{ij} - 2\lambda h b_{ij}(N) + \lambda^2 c_{ij} \quad (11.6)$$

where  $c_{ij}$  is a continuous function of  $u$  in  $D$ .

Then we have

$$\det(g_{ij}^\lambda) = a_0 + a_1\lambda + a_2\lambda^2 \quad (11.7)$$

with  $a_0 = \det g_{ij}$ ,  $a_1 = -2h(g_{11}b_{22}(N) + g_{22}b_{11}(N) - 2g_{12}b_{12}(N))$ , and  $a_2$  is a continuous function in  $u_1$ ,  $u_2$ , and  $\lambda$ .

Because  $S$  is regular, and the determinant function is continuous, we know that  $a_0$  has a positive minimum on  $cl(\Delta)$  (the closure of  $\Delta$ .) Then we can find an  $\epsilon$  such that  $|\lambda| < \epsilon$  means that  $\det(g_{ij}^\lambda) > 0$  on  $cl(\Delta)$ . Thus, for a small enough  $\epsilon$ , all surfaces  $S_\lambda$  restricted to  $\Delta$  are regular surfaces.

Now, looking at the Taylor series expansion of the determinant function, we get, for some positive constant  $M$ ,

$$|\sqrt{\det(g_{ij}^\lambda)} - (\sqrt{a_0} + \frac{a_1}{2\sqrt{a_0}})\lambda| < M\lambda^2 \quad (11.8)$$

Then, using the formula for the area of a surface, we have that the area of our original surface  $S$ ,  $A(0) = \int \int_\Delta \sqrt{a_0} du_1 du_2$ .

Integrating the equation with an  $M$  in it, we get

$$|A(\lambda) - A(0) - \lambda \int \int_\Delta \frac{a_1}{2\sqrt{a_0}} du_1 du_2| < M_1\lambda^2 \quad (11.9)$$

$$|\frac{A(\lambda) - A(0)}{\lambda} - \int \int_\Delta \frac{a_1}{2\sqrt{a_0}} du_1 du_2| < M_1\lambda. \quad (11.10)$$

Letting  $\lambda$  go to 0, and using  $H(N) = \frac{g_{22}b_{11}(N) + g_{11}b_{22}(N) - 2g_{12}b_{12}(N)}{2\det(g_{ij})}$ , we get

$$A'(0) = -2 \int \int_\Delta H(N)h(u)\sqrt{\det g_{ij}} du_1 du_2 (*) \quad (11.11)$$

or when integrating with respect to surface area

$$A'(0) = -2 \int \int_\Delta H(N)h(u)dA \quad (11.12)$$

□

From here it is clear that if  $H(N)$  is zero everywhere, then  $A'(0)$  is zero, and thus we have a critical point (hence minimal surfaces being misnamed: we can only ensure that  $A$  has a critical point by setting  $H(N)$  to zero everywhere.) Now we show the converse:

**Corollary 11.1.2.** *If  $S$  minimizes area, then its mean curvature vanishes everywhere.*

*Proof.* : Suppose the mean curvature doesn't vanish. Then there's some point  $a$  and a normal  $N(a)$  where  $H(N) \neq 0$  (we can assume  $H(N) > 0$  by choosing an appropriately oriented normal.) Then, with Lemma 2.2 from Osserman, we can find a neighborhood  $V_1$  of  $a$  where  $N$  is normal to  $S$ . This implies that there's a smaller neighborhood  $V_2$  contained in  $V_1$  where the mean curvature  $H(N)$  is positive. Now choose a function  $h$  which is positive on  $V_2$  and 0 elsewhere. Then the integral in (\*) is strictly positive, and thus  $A'(0)$  is strictly negative.

If  $V_2$  is small enough (contained in  $\Delta$ ), then on the boundary  $\gamma$ ,  $x(u) = y(u)$  for the original surface  $S$  and a surface  $S_\lambda$  respectively. Assuming that  $S$  minimizes area says that for all  $\lambda$ ,  $A(\lambda) \geq A(0)$ , which implies  $A'(0) = 0$  which is a contradiction since  $A'(0)$  was just shown to be strictly negative.  $\square$

## 11.2 Complex Functions

Working in  $\mathbb{C}$  is way different than working in just  $\mathbb{R}^2$ . For example: a complex function of a complex variable (i.e.,  $f : \mathbb{C} \rightarrow \mathbb{C}$ ) is called analytic if it is differentiable, and it can be shown that any analytic function is infinitely differentiable! It's pretty crazy.

I don't think we're going to show that in this class, though. But let's talk about derivatives of complex functions. They're defined in the same way as for real functions, i.e. the derivative of a function  $f$  (with either a real or complex variable  $x$ ) at a point  $a$  is

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}. \quad (11.13)$$

These derivatives work like we expect them to (things like the product rule, quotient rule, and chain rule are still valid.) But, there is a fundamental difference between considering a real variable and a complex variable.

**Exercise 4.** *Let  $f(z)$  be a real function of a complex variable ( $f : \mathbb{C} \rightarrow \mathbb{R}$ .) What can we say about  $f'(a)$  for any point  $a$ ?*

## 11.3 Analytic Functions and the Cauchy-Riemann Equations

So we defined earlier what an analytic function was, but I'll restate it here:

**Definition 11.3.1.** *A function  $f : \mathbb{C} \rightarrow \mathbb{C}$  is called analytic (or holomorphic, equivalently) if its first derivative exists where  $f$  is defined.*

We can also represent a complex function  $f$  by writing  $f(z) = u(z) + iv(z)$ , where  $u$  and  $v$  are real-valued.

When we look at the derivative of  $f$  at  $a$  :

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \quad (11.14)$$

we know that the limit as  $h$  approaches 0 must agree from all directions. So if we look at  $f'(a)$  as  $h$  approaches 0 along the real line (keeping the imaginary part of  $h$  constant), our derivative is a partial with respect to  $x$  and we get:

$$f'(z) = \frac{\delta f}{\delta x} = \frac{\delta u}{\delta x} + i \frac{\delta v}{\delta x} \quad (11.15)$$

Similarly, taking purely imaginary values for  $h$ , we get that

$$f'(z) = \lim_{k \rightarrow 0} \frac{f(z + ik) - f(z)}{k} = -i \frac{\delta f}{\delta y} = -i \frac{\delta u}{\delta y} + \frac{\delta v}{\delta y} \quad (11.16)$$

So we get that

$$\frac{\delta f}{\delta x} = -i \frac{\delta f}{\delta y} \quad (11.17)$$

and comparing real parts and imaginary parts,

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}, \quad \frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x} \quad (11.18)$$

These are the **Cauchy-Riemann equations**, and any analytic function must satisfy them.

## 11.4 Harmonic Functions

We assume that the functions  $u$  and  $v$  (given some analytic  $f = u + iv$ ) have continuous partial derivatives of all orders, and that the mixed partials are equal (this follows from knowing the derivative of an analytic function is itself analytic, as raved about earlier.) Then, using equality of mixed partials and the Cauchy-Riemann equations we can show that:

$$\Delta u = \frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0 \quad (11.19)$$

and

$$\Delta v = \frac{\delta^2 v}{\delta x^2} + \frac{\delta^2 v}{\delta y^2} = 0 \quad (11.20)$$

Defining any function  $f$  which satisfies *Laplace's equation*  $\Delta f = 0$  to be **harmonic**, we get that the real and imaginary parts of an analytic function are harmonic.

Conversely, say we have two harmonic functions  $u$  and  $v$ , and that they satisfy the Cauchy-Riemann equations (here  $v$  is called the **conjugate har-**

**monic function** of  $u$ .) We want to show that  $f = u + iv$  is analytic. This is done in the next lecture (Kai's Monday 10/18 lecture!)