

# Chapter 13

## Review on Complex Analysis II

(This lecture was given Friday, October 22, 2004.)

### 13.1 Poles and Singularities

(Following Ahlfors pages 127 - 129.)

We consider a function  $f(z)$  analytic in a neighborhood of  $a$ , except perhaps at  $a$  itself. (So  $f(z)$  is analytic on a region  $0 < |z - a| < \delta$ .)

**Definition 13.1.1.** *The number  $a$ , as above, is called an isolated singularity of  $f$ .*

We call  $a$  a *removable singularity* if we can simply define  $f(a)$  appropriately so that  $f(z)$  becomes analytic in the entire disk  $|z - a| < \delta$ . This is discussed in Ahlfors, page 124.

**Definition 13.1.2.** *If  $\lim_{z \rightarrow a} f(z) = \infty$ ,  $a$  is a pole of  $f(z)$ .*

With the case of a pole, we set  $f(a) = \infty$ . Then there exists a  $\delta' \leq \delta$  such that  $f(z) \neq 0$  on  $0 < |z - a| < \delta'$ . On this disk, we can look at  $g(z) = 1/f(z)$ , which is analytic on this disk, and more importantly has a removable singularity at  $a$ . So we can set  $g(a) = 0$ .

Now  $g(z)$  doesn't vanish identically, so we know that the zero of  $g$  at  $a$  has finite order and we can write  $g(z) = (z - a)^h g_h(z)$ , where  $g_h(z) \neq 0$  and is analytic (we can do this because an analytic function behaves locally like a polynomial. Since our function isn't identically zero, we can find a derivative  $g^{(h)}(a)$  that doesn't vanish, and then look at the Taylor series expansion from that term on, factoring out  $(z - a)^h$  from each term. See Kai's previous notes, Eq. 60.) We call  $h$  the *order* of the pole, and we can now write  $f(z) = (z - a)^{-h} f_h(z)$ , where  $f_h(z) = 1/g_h(z)$  is analytic and non-zero in a neighborhood of  $a$ .

**Definition 13.1.3.** : *A function  $f(z)$  analytic in a region  $\Omega$ , except for at its poles, is called meromorphic in  $\Omega$ .*

Equivalently, for every  $a \in \Omega$ , there is either a neighborhood  $|z - a| < \delta$  where the function is analytic, or else  $f(z)$  is analytic on  $0 < |z - a| < \delta$  and the isolated singularity there is a pole. So that means that the poles of a meromorphic function are isolated by definition. (What would happen if the poles weren't isolated?)

Looking at the quotient  $f(z)/g(z)$  of two analytic functions in  $\Omega$ , and assuming that  $g(z)$  isn't identically zero, we get a meromorphic function in  $\Omega$ . The possible poles here are the zeroes of  $g(z)$ , but a common zero of  $f(z)$  and  $g(z)$  could be a removable singularity ( $f(z) = (z^2 - 1)/(z + 1)$ , for example.) Similarly, the sum, product, and quotient of meromorphic functions are again meromorphic. When regarding the quotient of meromorphic functions, we exclude the possibility of the denominator being identically zero (otherwise, we'd have to consider  $f(z) = \infty$  to be a meromorphic function.)

Let's now take a deeper look at isolated singularities. Consider the conditions

$$\lim_{z \rightarrow a} |z - a|^\alpha |f(z)| = 0 \tag{13.1}$$

$$\lim_{z \rightarrow a} |z - a|^\alpha |f(z)| = \infty \tag{13.2}$$

where  $\alpha$  is a real number. If (1) holds for a given value of  $\alpha$ , then it holds for all larger values of  $\alpha$ , and thus we can find some integer  $m$  where it is true. This means that  $g(z) = (z - a)^m f(z)$  has a removable singularity and vanishes on  $a$ . From here, we know that either  $f(z)$  is identically zero (and then (1) holds for all  $\alpha$ ), or  $g(z) = (z - a)^m f(z)$  has a zero of finite order  $k$ . In the latter case, we can write  $g(z) = (z - a)^k (z - a)^{m-k} f(z)$ , where  $(z - a)^{m-k} f(z)$  is analytic. So if  $\alpha > h = m - k$ , (1) holds, and if  $\alpha < h$  (2) holds.

Now we assume that condition (2) holds for some  $\alpha$ . Then it holds for all smaller  $\alpha$ , and likewise for some integer  $n$ . The function  $g(z) = (z - a)^n f(z)$  has a pole of finite order  $l$ , and setting  $h = n + l$  (since now we write  $g(z) = (z - a)^{-l} (z - a)^{l+n} f(z)$ , where  $(z - a)^{l+n} f(z)$  is analytic) we find that condition (1) holds when  $\alpha > h$  and condition (2) holds when  $\alpha < h$ .

This means that given an isolated singularity, we have three cases to examine:

- i)  $f(z)$  is identically zero
- ii) there exists an integer  $h$  such that (1) holds for  $h > \alpha$  and (2) holds for  $h < \alpha$
- iii) neither (1) nor (2) holds for any  $\alpha$ .

Case i) is not interesting.

With case ii), we call  $h$  the *algebraic order* of  $f(z)$  at  $a$ . For a pole this is positive, for a zero it is negative, and it is zero when  $f(z)$  is analytic but not equal to zero at  $a$ . The algebraic order is always an integer – there is no single-valued analytic function which tends to 0 or  $\infty$  like a fractional power of  $|z - a|$

For case iii),  $a$  is called an *essential isolated singularity*. So in any neighborhood of an essential isolated singularity,  $f(z)$  is both unbounded and comes arbitrarily close to zero. This is illustrated by:

**Theorem 13.1.4 (Weierstrass).** *An analytic function comes arbitrarily close to any complex value in every neighborhood of an essential singularity.*

*Proof.* : Suppose that isn't true. Then we can find a complex number  $A$  and a  $\delta > 0$  such that  $|f(z) - A| > \delta$  in a neighborhood of  $a$  (except at  $z = a$ .) For any  $\alpha < 0$  we then have  $\lim_{z \rightarrow a} |z - a|^\alpha |f(z) - A| = \infty$ . So  $a$  would not be an essential singularity of  $f(z) - A$ .

Then we can find a  $\beta$  where  $\lim_{z \rightarrow a} |z - a|^\beta |f(z) - A| = 0$  (since we're now looking at a case ii) singularity) and we're free to choose  $\beta > 0$ . Then since in that case  $\lim_{z \rightarrow a} |z - a|^\beta = 0$ , it follows that  $\lim_{z \rightarrow a} |z - a|^\beta |f(z)| = 0$ , contradicting the fact that  $a$  is an essential singularity of  $f(z)$ .  $\square$