Chapter 15

Bernstein's Theorem

15.1 Minimal Surfaces and isothermal parametrizations

Note: This section will not be gone over in class, but it will be referred to.

Lemma 15.1.1 (Osserman 4.4). *Let S be a minimal surface. Every regular point p of S has a neighborhood in which there exists of reparametrization of S in terms of isothermal parameters.*

Proof. By a previous theorem (not discussed in class) there exists a neighborhood of the regular point which may be represented in a non-parametric form. Then we have that $x(x_1, x_2) = (x_1, x_2, f_3(x_1, x_2), \ldots, f_n(x_1, x_2))$. Defining $f = (f_3, f_4, \ldots, f_n)$, we let $p = \frac{\partial f}{\partial x_1}$, $q = \frac{\partial f}{\partial x_2}$, $r = \frac{\partial^2 f}{\partial x_1^2}$, $s = \frac{\partial^2 f}{\partial x_1 \partial x_2}$, and $t = \frac{\partial^t f}{\partial x_2^2}$. Last, we let $W = \sqrt{\det g_{ij}} = \sqrt{1 + |p|^2 + |q|^2 + |p|^2|q|^2 - (p \cdot q)^2}$. We then have (from last lecture)

$$\frac{\partial}{\partial x_1}\left(\frac{1 + |q|^2}{W}\right) = \frac{\partial}{\partial x_2}\left(\frac{p \cdot q}{W}\right)$$

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Then there exists a function $F(x_1, x_2)$ such that $\frac{\partial F}{\partial x_1} = \frac{1+|p|^2}{W}$ and $\frac{\partial F}{\partial x_2} = \frac{p \cdot q}{W}$. Why? Think back to 18.02 and let $V = (\frac{1+|p|^2}{W}, \frac{p \cdot q}{W}, 0)$ be a vector field in $\mathbb{R}^3$; then $| \bigtriangledown \times V| = \frac{\partial}{\partial x_2} \frac{1+|p|^2}{W} - \frac{\partial}{\partial x_1} \frac{p \cdot q}{W} = 0$, so there exists a function F such that $\bigtriangledown F = V$, which is the exact condition we wanted (once we get rid of the third dimension). Similarly there exists a function $G(x_1, x_2)$ with $\frac{\partial G}{\partial x_1} = \frac{p \cdot q}{W}$ and $\frac{\partial G}{\partial x_2} = \frac{\partial 1+|q|^2}{\partial W}$.

We now define $\xi(x_1, x_2)(x_1 + F(x_1, x_2), x_2 + G(x_1, x_2))$. We then find that $\frac{\partial \xi_1}{\partial x_1} = 1 + \frac{1+|p|^2}{W}$, $\frac{\partial \xi_2}{\partial x_2} = 1 + \frac{1+|q|^2}{W}$, and $\frac{\partial \xi_1}{\partial x_2} = \frac{\partial \xi_2}{\partial x_1} = \frac{p \cdot q}{W}$. Then (recalling the defintion of W^2) we can find that the magnitude of the Jacobian $\frac{\partial(\xi_1, \xi_2)}{\partial(x_1, x_)}$ is $2 + \frac{2+|p|^2+|q|^2}{W} > 0$. This implies that the transformation ξ has a local inverse $\hat{x}$ at p. Judicial use of the inverse function theorem will show that with respect to the parameters ξ_1 and ξ_2, $g_{11} = g_{22}$ and $g_{12} = 0$, so these are isothermal coordinates; see Osserman p 32 for details. □

We also have the following result:

Lemma 15.1.2 (Osserman 4.5). *Let a surface S be defined by an isothermal parametrization $x(u)$, and let $\widetilde{S}$ be a reparametrization of S defined by a diffeomorphism with matrix U. Then $\tilde{u_1}$, $\tilde{u_2}$ are isothermal parameters if and only if the map U is either conformal or anti-conformal.*

Proof. For a map to be conformal or anti-conformal means that it preserves $|\theta|$, or alternatively that it preserves $\cos\theta$. (It also needs to be continuous enough that it isn't flipping the sign back and forth.) If U is a constant μ times an orthogonal matrix, then $\mu|v| = |Uv|$ for all v since $\mu^2 \langle v, v \rangle = \langle Uv, Uv \rangle$; thus if θ is the angle between vectors v and w and θ' is the angle between Uv and Uw, we have that $\cos\theta = \frac{\mu^2 \langle v,w \rangle}{\mu^2 |v||w|} = \frac{\langle Uv, Uw \rangle}{|Uv||Uw|} = \cos\theta'$. So for diffeomorphisms with matrix U, U being conformal or anti-conformal is equivalent to U being a constant multiple of an orthogonal matrix.

Now, since x is isothermal, we have that $g_{ij} = \lambda^2 \delta_{ij}$ (where δ_{ij} is the Kronecker delta). By a theorem on page 5 about change of coordinates, we know that $\widetilde{G} = U^T G U = \lambda^2 U^T U$. So $\tilde{u_1}$, $\tilde{u_2}$ is isothermal iff $\tilde{g}_{ij} = \tilde{\lambda}^2 \delta_{ij}$, which is to say that $I = \frac{\lambda^2}{\tilde{\lambda}^2} U^T U$, which is to say that $\frac{\tilde{\lambda}}{\lambda} U$ is orthogonal. But we have already shown that this is equivalent to U being conformal or anti-conformal. □

15.2 Bernstein's Theorem: Some Preliminary Lemmas

The main goal of today is to prove Bernstein's Theorem, which has the nice corollary that in $\mathbb{R}^3$, the only minimal surface that is defined in non-parametric form on the entire x_1, x_2 plane is a plane. This makes sense: the catenoid and helicoid are not going to give you nonparametric forms since no projection of them is injective, and Scherk's surface may be nonparametric but it's only defined on a checkerboard. We have a bunch of lemmas to work through first.

Lemma 15.2.1 (Osserman 5.1). *Let $E \colon D \to \mathbb{R}$ be a C^2 function on a convex domain D, and suppose that the Hessian matrix $\left(\frac{\partial^2 E}{\partial x_i \partial x_j}\right)$ evaluated at any point is positive definite. (This means that the quadratic form it defines sends every nonzero vector to a positive number, or equivalently that it is symmetric with positive eigenvalues.) Define a mapping $\phi \colon D \to \mathbb{R}^2$ with $\phi(x_1, x_2) = (\frac{\partial E}{\partial x_1}(x_1, x_2), \frac{\partial E}{\partial x_2}(x_1, x_2))$ (since $\frac{\partial E}{\partial x_1} \colon D \to \mathbb{R}$). Let a and b be distinct points of D; then $(b - a) \cdot (\phi(b) - \phi(a)) > 0$.*

Proof. Let $G(t) = E(tb + (1-t)a) = E(tb_1 + (1-t)a_1, tb_2 + (1-t)b_2)$ for $t \in [0, 1]$. Then

$$G'(t) = \sum_{i=1}^{2} \left(\frac{\partial E}{\partial x_i}(tb + (1-t)a) \right) (b_i - a_i)$$

(note that the $tb + (1-t)a$ here is the *argument* of $\frac{\partial E}{\partial x_i}$, not a multiplied factor) and

$$G''(t) = \sum_{i,j=1}^{2} \left(\frac{\partial^2}{\partial x_i \partial x_j}(tb + (1-t)a) \right) (b_i - a_i)(b_j - a_j)$$

But this is just the quadratic form of $\left(\frac{\partial^2 E}{\partial x_i \partial x_j}\right)$ evaluated at the point $tb + (1-t)a$, applied to the nonzero vector $b - a$. By positive definiteness, we have that $G''(t) > 0$ for $t \in [0,1]$. So $G'(1) > G'(0)$, which is to say that $\sum \phi(b)_i(b_i - a_i) > \sum \phi(a)_i(b_i - a_i)$, which is to say that $(\phi(b) - \phi(a)) \cdot (b - a) > 0$. □

Lemma 15.2.2 (Osserman 5.2). *Assume the hypotheses of Osserman Lemma 5.1. Define the map* $z\colon D \to \mathbb{R}^2$ *by* $z_i(x_1, x_2) = x_i + \phi_i(x_1, x_2)$*. Then given distinct points* $a, b \in D$*, we have that* $(z(b) - z(a)) \cdot (b - a) > |b - a|^2$*, and* $|z(b) - z(a)| > |b - a|$*.*

Proof. Since $z(b) - z(a) = (b - a) + (\phi(b) - \phi(a))$, we have that $(z(b) - z(a)) \cdot (b - a) = |b - a|^2 + (\phi(b) - \phi(a)) \cdot (b - a) > |b - a|^2$ by the previous lemma.

Then $|b - a|^2 < |(z(b) - z(a)) \cdot (b - a)| \leq |z(b) - z(a)||b - a|$, where the second inequality holds by Cauchy-Schwarz; so $|b - a| < |z(b) - z(a)|$. □

Lemma 15.2.3 (Osserman 5.3). *Assume the hypotheses of Osserman Lemma 5.2. If* D *is the disk* $x_1^2 + x_2^2 < R^2$*, then the map* z *is a diffeomorphism of* D *onto a domain* Δ *which includes a disk of radius* R *around* $z(0)$*.*

Proof. We know that z is continuously differentiable, since $E \in C^2$. If $x(t)$ is any differentiable curve in D and $z(t)$ is its image under z, then it follows from the previous lemma that $|z'(t)| > |x'(t)|$; thus the determinant of the matrix dz (which is to say, the Jacobian) is greater than 1, since $z'(t) = (dz)x'(t)$ implies that $|z'(t)| = \det dz|x'(t)|$. So since the Jacobian is everywhere greater than 1, the map is a local diffeomorphism everywhere. It's also injective (because

$\phi(b) - \phi(a) = 0$ implies that $b - a = 0$ by the previous lemma), so it's in fact a (global) diffeomorphism onto a domain Δ.

We must show that Δ includes all points z such that $z - z(0) < R$. If Δ is the whole plane this is obvious; otherwise there is a point Z in the complement of Δ (which is closed) which minimizes the distance to $z(0)$. Let Z^k be a sequence of points in $\mathbb{R}^2 - \Delta$ which approach Z (if this didn't exist, we could find a point in $\mathbb{R}^2 - \Delta$ closer to $z(0)$ than Z), and since z is a diffeomorphism, we let x^k be the sequence of points mapped onto Z^k by z. The points x^k cannot have a point of accumulation in D, since that would be mapped by z onto a point of Δ, and we are assuming that $Z \notin \Delta$. But x^k must have an accumulation point in $\mathbb{R}^2$ in order for their image to; so $|x^k| \to R$ as $k \to \infty$; since $|\mathbb{Z}^k - z(0)| > |x^k - 0|$ by the previous lemma, we have that $|Z - z(0)| \geq R$, so every point within R of $z(0)$ is in Δ. □

Lemma 15.2.4 (Osserman 5.4). *Let $f(x_1, x_2)$ be a non-parametric solution to the minimal surface equation in the disk of radius R around the origin. Then the map ξ defined earlier is a diffeomorphism onto a domain Δ which includes a disk of radius R around $\xi(0)$.*

Proof. It follows from the defining characteristics of F and G that there exists a function E satisfying $\frac{\partial E}{\partial x_1} = F$ and $\frac{\partial E}{\partial x_2} = G$, for the same reason that F and G exist. Then $E \in C^2$, and $\frac{\partial^2 E}{\partial x_1^2} = \frac{1+|p|^2}{W} > 0$, and $\det \frac{\partial^2 E}{\partial x_i \partial x_j} = \frac{\partial(F,G)}{\partial(x_1,x_2)} = 1 > 0$ (by the definition of W, it's a simple check). Any matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ with $a > 0$ and $ac - b^2 > 0$ must have $c > 0$, so its trace and determinant are both positive, so the sum and product of its eigenvalues are both positive, so it is positive definite. So the Hessian of E is positive definite. We can see that the mapping z defined in (our version of) Osserman Lemma 5.2 is in this case the same map as ξ defined in (our version of) Osserman Lemma 4.4. So by Osserman Lemma 5.3, we have that ξ is a diffeomorphism onto a domain Δ which includes a disk of radius R around $\xi(0)$. □

Lemma 15.2.5 (Osserman 5.5). *Let $f\colon D \to \mathbb{R}$ be a C^1 function. Then the surface S in $\mathbb{R}^3$ defined in non-parametric form by $x_3 = f(x_1, f_2)$ lies on a plane iff there exists a nonsingular linear transformation $\psi\colon U \to D$ from some domain U such that u_1, u_2 are isothermal parameters on S.*

Proof. Suppose such parameters u_1, u_2 exist. Letting $\phi_k(\zeta) = \frac{\partial x_k}{\partial u_1} - i\frac{\partial x_k}{\partial u_2}$, for $1 \leq k \leq 3$, we see that ϕ_1 and ϕ_2 are constant because x_1 and x_2 are linear in u_1 and u_2. We know from a previous lecture that u_1 and u_2 are isothermal parameters iff $\sum_{k=1}^{3} \phi_k^2(\zeta)$ is zero for all ζ, so ϕ_3 is constant too. (Well, it implies that ϕ_3^2 is constant, which constrains it to at most two values, and since ϕ_3 must be continuous, it must be constant.) This means that x_3 has a constant gradient with respect to u_1, u_2 and thus also with respect to x_1, x_2. This means that we must have $f(x_1, x_2) = Ax_1 + Bx_2 + C$; but this is the equation of a plane.

Conversely, if $f(x_1, x_2)$ is a part of a plane, then it equals $Ax_1 + Bx_2 + C$ for some constants A, B, and C. Then the map $x(u_1, u_2) = (\lambda A u_1 + B u_2, \lambda B u_1 - A u_2)$ with $\lambda^2 = \frac{1}{1+A^2+B^2}$ is isothermal. To check this, we see that $\phi_1 = \lambda A - iB$, $\phi_2 = \lambda B + iA$, $\phi_1^2 = \lambda^2 A^2 - B^2 - 2\lambda ABi$, $\phi_2^2 = \lambda^2 B^2 - A^2 + 2\lambda ABi$. $x_3 = Ax_1 + Bx_2 + C = A(\lambda A u_1 + B u_2) + B(\lambda B u_1 - A u_2) + C$, so $\phi_3 = \lambda(A^2 + B^2)$ and $\phi^2 = \lambda^2(A^2+B^2)^2$. Then $\phi_1^2 + \phi_2^2 + \phi_3^2 = \lambda^2(A^2+B^2) - (A^2+B^2) + \lambda^2(A^2 + B^2)^2 = (A^2+B^2)(\lambda^2 - 1 + \lambda^2(A^2+B^2)) = (A^2+B^2)(\lambda^2(1+A^2+B^2) - 1) = (A^2+B^2)(1-1) = 0$, so this is isothermal. □

15.3 Bernstein's Theorem

Theorem 15.3.1 (Bernstein's Theorem, Osserman 5.1). *Let $f(x_1, x_2)$ be a solution of the non-parametric minimal surface equation defined in the entire x_1, x_2 plane. Then there exists a nonsingular linear transformation $x_1 = u_1$, $x_2 = au_1 + bu_2$ with $b > 0$ such that u_1, u_2 are isothermal parameters on the entire u-plane for the minimal surface S defined by $x_k = f_k(x_1, x_2)$ $(3 \leq k \leq n)$.*

Proof. Define the map ξ as in our version of Osserman Lemma 4.4. Osserman Lemma 5.4 shows that this is a diffeomorphism from the entire x-plane onto the entire ξ-plane. We know from Osserman Lemma 4.4 that ξ is a set of isothermal parameters on S. By Osserman Lemma 4.3 (which Nizam proved), the functions $\phi_k(\zeta) = \frac{\partial x_k}{\partial \xi_1} - i\frac{\partial x_k}{\partial \xi_2}$ $(1 \leq k \leq n)$ are analytic functions of ζ. We can see that $\Im(\bar{\phi_1}\phi_2) = -\frac{\partial(x_1,x_2)}{\partial(\xi_1,\xi_2)}$; since this Jacobian is always positive (see proof of Osserman Lemma 4.4), we can see that $\phi_1 \neq 0$, $\phi_2 \neq 0$, and that $\Im\frac{\phi_2}{\phi_1} = \frac{1}{|\phi_1|^2}\Im(\bar{\phi_1}\phi_2) < 0$. So the function $\frac{\phi_2}{\phi_1}$ is analytic on the whole ζ-plane and has negative imaginary part everywhere. By Picard's Theorem, an analytic function that misses more than one value is constant, so $\frac{\phi_2}{\phi_1} = C$ where $C = a - ib$. That is $\phi_2 = (a - ib)\phi_1$. The real part of this equation is $\frac{\partial x_2}{\partial \xi_1} = a\frac{\partial x_1}{\partial \xi_1} - b\frac{\partial x_1}{\partial \xi_2}$, and the imaginary part is $\frac{\partial x_2}{\partial \xi_2} = b\frac{\partial x_1}{\partial \xi_1} + a\frac{\partial x_1}{\partial \xi_2}$. If we then apply the linear transformation from the statement of the theorem, using the a and b that we have, this becomes $\frac{\partial u_1}{\partial \xi_1} = \frac{\partial u_2}{\partial \xi_2}$ and $\frac{\partial u_2}{\partial \xi_2} = -\frac{\partial u_1}{\partial \xi_2}$: the Cauchy-Reimann equations! So $u_1 + iu_2$ is an analytic function of $\xi_1 + i\xi_2$. But by Osserman Lemma 4.5, this implies that u_1, u_2 are also isothermal parameters, which proves the theorem. □

This (with Osserman Lemma 5.5) has the immediate corollary that for $n = 3$, the only solution of the non-parametric minimal surface equation on the entire x-plane is surface that is a plane. This gives us a nice way to generate lots of weird minimal surfaces in dimensions 4 and up by starting with analytic functions; this is Osserman Corollary 3, but I do not have time to show this.