Chapter 17

Complete Minimal Surfaces

Reading:

- Osserman [7] Pg. 49-52,
- Do Carmo [2] Pg. 325-335.

17.1 Complete Surfaces

In order to study regular surfaces globally, we need some global hypothesis to ensure that the surface cannot be extended further as a regular surface. Compactness serves this purpose, but it would be useful to have a weaker hypothesis than compctness which could still have the same effect.

Definition 17.1.1. *A regular (connected) surface S is said to be* **extendable** *if there exists a regular (connected) surface $\bar{S}$ such that $S \subset \bar{S}$ as a proper subset. If there exists no such $\bar{S}$, then S is said to be* **nonextendable**.

Definition 17.1.2. *A regular surface S is said to be* **complete** *when for every point $p \in S$, any parametrized geodesic $\gamma : [0, \epsilon) \to S$ of S, starting from $p = \gamma(0)$, may be extended into a parametrized geodesic $\bar{\gamma} : \mathbf{R} \to S$, defined on the entire line $\mathbf{R}$.*

Example 11 (Examples of complete/non-complete surfaces). *The following are some examples of complete/non-complete surfaces.*

1. *The plane is a complete surface.*

2. *The cone minus the vertex is a noncomplete surface, since by extending a generator (which is a geodesic) sufficiently we reach the vertex, which does not belong to the surface.*

3. *A sphere is a complete surface, since its parametrized geodesics (the great circles) may be defined for every real value.*

4. *The cylinder is a complete surface since its geodesics (circles, lines and helices) can be defined for all real values*

5. *A surface $S - \{p\}$ obtained by removing a point p from a complete surface S is not complete, since there exists a geodesic of $S - \{p\}$ that starts from a point in the neighborhood of p and cannot be extended through p.*

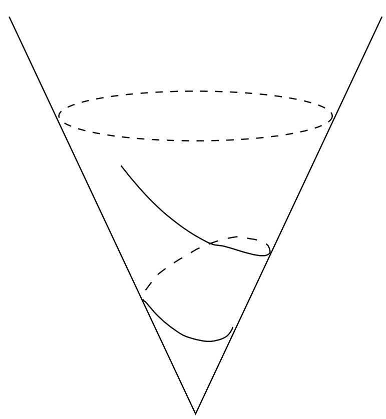

Figure 17.1: A geodesic on a cone will eventually approach the vertex

Proposition 17.1.3. *A complete surface S is nonextendable.*

Proof. Let us assume that S is extendable and obtain a contradiction. If S is extendable, then there exists a regular (connected) surface $\bar{S}$ such that $S \subset \bar{S}$. Since S is a regular surface, S is open in $\bar{S}$. The boundary $\mathrm{Bd}(S)$ of S is nonempty, so there exists a point $p \in \mathrm{Bd}(S)$ such that $p \notin S$.

Let $\bar{V} \subset \bar{S}$ be a neighborhood of p in $\bar{S}$ such that every $q \in \bar{V}$ may be joined to p by a unique geodesic of $\bar{S}$. Since $p \in \mathrm{Bd}(S)$, some $q_0 \in \bar{V}$ belongs to S. Let $\bar{\gamma} : [0,1] \to \bar{S}$ be a geodesic of $\bar{S}$, with $\bar{\gamma}(0) = p$ and $\bar{\gamma}(1) = q_0$. It is clear that $\alpha : [0,\epsilon) \to S$, given by $\alpha(t) = \bar{\gamma}(1-t)$, is a geodesic of S, with $\alpha(0) = q_0$, the extension of which to the line $\mathbf{R}$ would pass through p for $t = 1$. Since $p \notin S$, this geodesic cannot be extended, which contradicts the hypothesis of completness and concludes the proof. □

Proposition 17.1.4. *A closed surface $S \subset \mathbf{R}^3$ is complete*

Corollary 17.1.5. *A compact surface is complete.*

Theorem 17.1.6 (Hopf-Rinow). *Let S be a complete surface. Given two points $p, q \in S$, there exists a nimimal geodesic joining p to q.*

17.2 Relationship Between Conformal and Complex-Analytic Maps

In surfaces, conformal maps are basically the same as complex-analytic maps. For this section, let $U \subset \mathbf{C}$ be a open subset, and $z \in U$.

Definition 17.2.1. *A function $f : U \to \mathbf{C}$ is* **conformal** *if the map df_z preserves angle and sign of angles.*

Proposition 17.2.2. *A function $f : U \to \mathbf{C}$ is conformal at $z \in U$ iff f is a complex-analytic function at z and $f'(z) \neq 0$.*

Proof. Let B be the matrix representation of df_z in the usual basis. Then f is conformal $\Leftrightarrow B = cA$ where $A \in SO(2)$ and $c > 0$. Thus

$$BB^T = c^2 I \quad \Leftrightarrow \quad B^T = (\det B)B^{-1} \tag{17.1}$$

Let $z = x + iy$ and $f(z) = f(x, y) = u(x, y) + iv(x, y)$, then

$$B = \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} \tag{17.2}$$

where $u_y = \frac{\partial u}{\partial y}$. However, from Eq. 17.1, we have

$$\begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} = \begin{pmatrix} v_y & -u_y \\ -v_x & u_x \end{pmatrix} \tag{17.3}$$

which implies the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x. \tag{17.4}$$

Thus f is complex-analytic. □

17.3 Riemann Surface

Definition 17.3.1. *A* **Riemann Surface** *M is a 1-dim complex analytic manifold, i.e. each $p \in M$ has a neighborhood which is homeomorphic to a neighborhood in* **C**, *and the transition functions are complex analytic.*

In order to study Riemann surface, one needs to know the basic of harmonic and subharmonic functions.

Table 17.1: The analogues of harmonic and subharmonic functions on **R**

$\mathbb{R}$	$\mathbb{C}$
Linear	Harmonic
Convex	subharmonic

Definition 17.3.2. *A function $h : \mathbf{R} \to \mathbf{R}$ is* **harmonic** *iff it is in the form $h(x) = ax + b$, where $a, b \in \mathbf{R}$. In other words, $\Delta h = 0$ where $\Delta = \frac{d^2}{dx^2}$.*

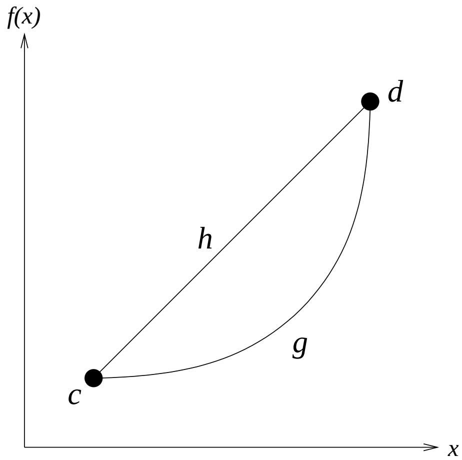

Figure 17.2: A graphical representation of a harmonic function h and a subhamonic function g in $\mathbb{R}$.

Definition 17.3.3. *A function* $g : \mathbf{R} \to \mathbf{R}$ *is* **convex** *if for every interval* $[c, d] \subset \mathbf{R}$, $g(x) < h(x)$ *for* $x \in (c, d)$ *where* h *is the linear function such that* $h(c) = g(c)$ *and* $h(d) = g(d)$.

Definition 17.3.4 (Second definition of convex functions). *If* $g : \mathbf{R} \to \mathbf{R}$ *is* **convex** *and* $g \leq \tilde{h}$ *on* (c, d) *for* $\tilde{h}$ *a harmonic function, then either* $g < \tilde{h}$ *or* $g \equiv \tilde{h}$ *there.*

Subharmonic functions on $\mathbb{C}$ are just the equivalents of convex functions on $mathbbR$.

Definition 17.3.5. *A function* $g : M \to \mathbb{R}$ *is* **subharmonic** *on a Riemann surface* M *if*

1. g *is constant.*

2. *For any domain* D *and any harmonic functions* $h : D \to \mathbb{R}$, *if* $g \leq h$ *on* D, *then* $g < h$ *on* D *or* $g = h$ *on* D.

3. *The difference* $g - h$ *satisfies the maximum principle on* D, *i.e.* $g - h$ *cannot have a maximum on* D *unless it is constant.*

Definition 17.3.6. *A Riemann surface M is* **hyperbolic** *if it supports a non-constant negative subharmonic function.*

Note: *If M is compact, then all constant functions on M that satisfy the maximum principle are constant. Therefore M is not hyperbolic.*

Definition 17.3.7. *A Riemann surface M is* **parabolic** *if it is not compact nor hyperbolic.*

Theorem 17.3.8 (Koebe-Uniformization Theorem). *If M is a simply connected Riemann surface, then*

1. *if M is compact, M is conformally equivalent to the sphere.*

2. *if M is parabolic, M is conformally equivalent to the complex plane.*

3. *if M is hyperbolic, M is conformally equivalent to the unit disc on the complex plane. But note that the disc has a hyperbolic metric*

$$ds^2 = \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}. \qquad (17.5)$$

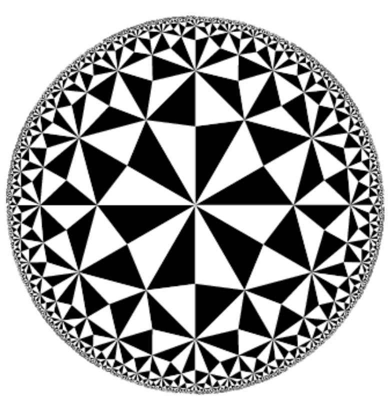

Figure 17.3: The Poincaré Hyperbolic Disk [9]

Table 17.2: Categorization of Riemann surfaces

Type	Conformally equivalent to	Remark
Hyperbolic	sphere	supports a non-constant negative subharmonic function
Compact	$\mathbb{C}$	
Parabolic	$D = \{z \in \mathbb{C} \| \|z\| < 1\}$	Not hyperbolic and not compact

17.4 Covering Surface

Definition 17.4.1. *A* **covering surface** *of a topological 2-manifold M is a topological 2-manifold $\tilde{M}$ and a map*

$$\rho : \tilde{M} \to M \tag{17.6}$$

such that ρ is a local homeomorphic map.

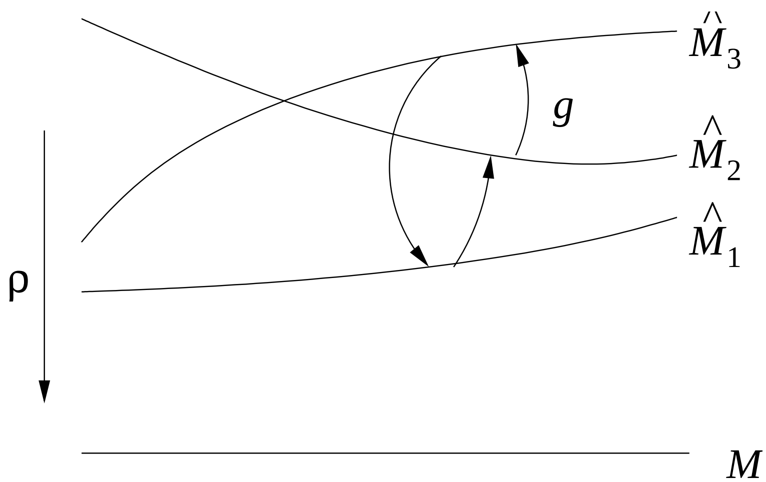

Figure 17.4: Covering surfaces

Definition 17.4.2. *A covering transformation of $\tilde{M}$ is a homeomorphism $g : \tilde{M} \to \tilde{M}$ such that $\rho \circ g = \rho$*

This forms a group G.

Proposition 17.4.3. *Every surface (2-manifold) M has a covering space $(\hat{M}, \rho)$ such that $\tilde{M}$ is simply connected, and*

$$\hat{M}/G \cong M \tag{17.7}$$