Chapter 19

Gauss Maps and Minimal Surfaces

19.1 Two Definitions of Completeness

We've already seen do Carmo's definition of a complete surface — one where every partial geodesic is extendable to a geodesic defined on all of $\mathbb{R}$. Osserman uses a different definition of complete, which we will show to be equivalent (this is also exercise 7 on page 336 of do Carmo).

A *divergent curve* on S is a differentiable map $\alpha\colon [0,\infty) \to S$ such that for every compact subset $K \subset S$ there exists a $t_0 \in (0,\infty)$ with $\alpha(t) \notin K$ for all $t > t_0$ (that is, α leaves every compact subset of S). We define the *length* of a divergent curve as $\lim_{t\to\infty} \int_0^t |\alpha'(t)|\,dt$, which can be unbounded. Osserman's definition of complete is that every divergent curve has unbounded length. We will sketch a proof that this is an equivalent definition.

First, we will assume that every divergent curve in S has unbounded length and show that every geodesic in S can be extended to all of $\mathbb{R}$. Let $\gamma\colon [0,\epsilon) \to S$ be a geodesic that cannot be extended to all of $\mathbb{R}$; without loss of generality assume specifically that it cannot of be extended to $[0,\infty)$. Then the set of numbers x such that γ can be extended to $[0,x)$ is nonempty (because it contains ϵ) and bounded above (because it cannot be extended to $[0, \infty)$), so it has an inf R. We note that since we can extend γ to $[0, R-\delta)$ for all (small) δ, we can in fact extend it to $\gamma' \colon [0, R) \to S$. Because γ' has constant speed, it must tend to a limit point $q \in \mathbb{R}^n$ (by completeness of $\mathbb{R}^n$ using a standard topological definition of completeness involving Cauchy sequences). Let $\alpha \colon [0, \infty) \to S$ be defined by $\alpha(t) = \gamma'(R(1 - e^{-t})$. Then α is just a reparametrization of γ', so it has the same length as γ', which is (because γ' is a geodesic) a constant multiple of R and thus bounded. So if we can show that α is a divergent curve, we will have a contradiction. Clearly q is also a limit point of α, since it is a reparametrization of γ'. If $q \in S$, then a regular neighborhood of q is contained in S and we could have extended the geodesic further, so $q \in Bd(S) - S$. So if α is in a compact (and thus closed) subset of S for arbitrarily large values of t, q must be in that set too, which is a contradiction. So in fact every geodesic can be extended to $\mathbb{R}$.

Next we assume that every geodesic can be extended to all of $\mathbb{R}$ and show that every divergent curve has unbounded length. Let α be a divergent curve with bounded length. Then we have for any $k > 0$ that $\lim_{n\to\infty} \int_n^{n+k} |\alpha'(t)|\, dt = 0$ — that is, points on α get arbitrarily close to each other, so because $\mathbb{R}^3$ is complete (in the Cauchy sense) α has a limit point q in $\mathbb{R}^3$. q cannot lie on S, because otherwise (the image under a chart of) a closed ball around q would be a compact set that α doesn't leave, and we know that α is divergent. So $q \notin S$. I don't quite see how to finish the proof here, but if it's true that if S is *any* surface (not just a complete one) then $S - \{p\}$ is not complete (in the geodesic sense), then this implies that our surface is not complete. I'm not sure if that's true though.

19.2 Image of S under the Gauss map

One very important consequence of the WERI representation is that the Gauss map $N \colon S \to S^2$ is just the function g, with S^2 the Riemann sphere;

that is, if $p\colon S^2 \to \mathbb{C}\cup\{\infty\}$ is stereographic projection, then $g = p\circ N$. Nizam proved this in his notes; the proof is mostly a matter of working through the algebra.

Lemma 19.2.1 (Osserman Lemma 8.4). *A minimal surface $S \subset \mathbb{R}^3$ that is defined on the whole plane is either a plane or has an image under the Gauss map that omits at most two points.*

Proof. We can find a WERI representation unless $\phi_1 = i\phi_2$ and $\phi_3 = 0$, but this means that x_3 is constant so that S is a plane. Otherwise, g is meromorphic in the entire plane, so by Picard's Theorem it takes on all values with at most two exceptions or is constant; so the Gauss map either takes on all values except for maybe two or is constant, and the latter case is a plane. □

Theorem 19.2.2 (Osserman Theorem 8.1). *Let S be a complete regular minimal surface in $\mathbb{R}^3$. Then S is a plane or the image of S under N is dense in the sphere.*

Proof. If the image is not everywhere dense then it omits a neighborhood of some point, which without loss of generality we can assume to be $N = (0, 0, 1)$. If we can prove that x is defined on the entire plane, then by the previous lemma we have our result. I do not entirely understand the proof, but it involves finding a divergent path of bounded length. □

We note that this implies Bernstein's Theorem, since a nonparametric minimal surface misses the entire bottom half of the sphere. So how many points can we miss?

Theorem 19.2.3 (Osserman Theorem 8.3). *Let E be an arbitrary set of $k \leq 4$ points on the unit sphere. Then there exists a complete regular minimal surface in $\mathbb{R}^3$ whose image under the Gauss map omits precisely the set E.*

Proof. We can assume (by rotation) that E contains the north pole $N = (0,0,1)$. If in fact $E = \{N\}$, then we can take $f(\zeta) = 1$ and $g(\zeta) = \zeta$, which clearly obey the properties that f and g must (as they are both analytic); since g takes on all values in $\mathbb{C}$, N must take on all values of $S^2 - \{N\}$ by inverse stereographic projection. (This is called Enneper's surface.) Otherwise let the points of $E - \{N\}$ correspond to points $w_m \in \mathbb{C}$ under stereographic projection. Let

$$f(\zeta) = \frac{1}{\prod(\zeta - w_m)}, g(\zeta) = \zeta$$

and use WERI with the domain $\mathbb{C} - \{w_1, \ldots, w_{k-1}\}$. Clearly g takes on all values except for the points w_m, so the image of the Gauss map omits only the values in E. f and g are both analytic (since the points where it looks like f would have poles are not in the domain). It remains to show that the surface is complete. We can show that in general the path length of a curve C equals $\int_C \frac{1}{2}|f|(1+|g|^2)|d\zeta|$. The only way a path can be divergent here is if it tends towards ∞ or one of the points w_m; in the former case the degree of $|f|(1+|g|^2)$ is at least -1 (because there are at most three terms on the bottom of f), so it becomes unbounded; in the latter case g goes to a constant and $|f|$ becomes unbounded, so every divergent curve has unbounded length and the surface is complete. □

It has been proven by Xavier (see p 149 of Osserman) that no more than six directions can be omitted, and as of the publication of Osserman it is not known whether five or six directions can be omitted.

19.3 Gauss curvature of minimal surfaces

Nizam showed that the Gauss curvature of a minimal surface depends only on its first fundamental form as $K = -\frac{1}{2g_{11}}\Delta(\ln g_{11})$; doing the appropriate calculations (starting with $g_{11} = 2|\phi|^2$ shows that we can write it in terms of

f and g as

$$K = -\left(\frac{4|g'|}{|f|(1+|g|^2)^2}\right)^2$$

This implies that the Gauss curvature of a minimal surface is non-positive everywhere (which is not surprising, since $K = k_1k_2 = -k_1^2$). It also implies that it can have only isolated zeros unless S is a plane. This is because K is zero precisely when the analytic (according to Osserman, though I don't see why) function g' has zeros, which is either isolated or everywhere. But if g' is identically zero, then g is constant, so N is constant, so S is a plane.

Consider, for an arbitrary minimal surface in $\mathbb{R}^3$, the following sequence of mappings:

$$D \xrightarrow{x(\zeta)} S \xrightarrow{N} S^2 \xrightarrow{p} \mathbb{C}$$

where p is stereographic projection onto the w-plane. The composition of all of these maps is g, as we have seen. Given a differentiable curve $\zeta(t)$ in D, if $s(t)$ is the arc length of its image on S, then (as mentioned above)

$$\frac{ds}{dt} = \frac{1}{2}|f|(1+|g|^2)|\frac{d\zeta}{dt}|$$

The arc length of the image in the w-plane is simply

$$abs\frac{dw}{dt} = |g'(\zeta)||\frac{d\zeta}{dt}|$$

because the composed map is g. If $\sigma(t)$ is arc length on the sphere, then by computation on the definition of stereographic projection we can show that

$$\frac{d\sigma}{dt} = \frac{2}{1+|w|^2}|\frac{dw}{dt}|$$

(note that $|w|$ here is the same as $|g|$. So dividing through we find that

$$\frac{\frac{d\sigma}{dt}}{\frac{ds}{dt}} = \frac{4|g'|}{|f|(1+|g|^2)^2} = \sqrt{|K|}$$

So there is a natural definition of Gauss curvature in terms of the Gauss map.

We define the total curvature of a surface to be the integral $\iint K$. We can show that this is in fact equal to the negative of spherical area of the image under the Gauss map, counting multiple coverings multiply.

19.4 Complete manifolds that are isometric to compact manifolds minus points

Theorem 19.4.1 (Osserman 9.1). *Let M be a complete Riemannian 2-manifold with $K \leq 0$ everywhere and $\iint |K| < \infty$. Then there exists a compact 2-manifold $\hat{M}$ and a finite set $P \subset \hat{M}$ such that M is isometric to $\hat{M} - P$.*

(Proof not given.)

Lemma 19.4.2 (Osserman 9.5). *Let x define a complete regular minimal surface S in $\mathbb{R}^3$. If the total curvature of $S$is finite, then the conclusion of the previous theorem holds and the function $g = p \circ N$ extends to a meromorphic function on $\hat{M}$.*

Proof. We already know that $K \leq 0$. This implies that $\iint |K| = |\iint K|$, the absolute value of the total curvature, which is finite. So the previous theorem holds. The only way that g could fail to extend is if it has an essential singularity at a point of P, but that would cause it to assume (almost) every value infinitely often, which would imply that the spherical area of the image of the Gauss map is infinite, which contradicts our assumption of finite total curvature. □

Theorem 19.4.3 (Osserman 9.2). *Let S be a complete minimal surface in $\mathbb{R}^3$. Then the total curvature of S is $-4\pi m$ for a nonnegative integer m, or $-\infty$.*

Proof. Since $K \leq 0$, either $\iint K$ diverges to $-\infty$, or it (the total curvature) is finite. Because K is preserved by isometries, we apply the previous lemma and see that the total curvature is the negative of the spherical area of the image under g of $\hat{M} - P$. Because g is meromorphic, it is either constant or takes on each value a fixed number of times m. So either the image is a single point (so the total curvature is $-4\pi 0$) or an m-fold cover of the sphere (so the total curvature is $-4\pi m$). $\square$