Chapter 4

Implicit Function Theorem

4.1 Implicit Functions

Theorem 4.1.1. ***Implicit Function Theorem*** *Suppose* $f : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ *is continuously differentiable in an open set containing* (a, b) *and* $f(a, b) = 0$. *Let* M *be the* $m \times m$ *matrix* $D_{n+j} f^i(a, b), 1 \leq i, j \leq m$ *If* $det(M) \neq 0$, *there is an open set* $A \subset \mathbb{R}^n$ *containing* a *and an open set* $B \subset \mathbb{R}^m$ *containing* b, *with the following property: for each* $x \in A$ *there is a unique* $g(x) \in B$ *such that* $f(x, g(x)) = 0$. *The function* g *is differentiable.*

proof Define $F : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n \times \mathbb{R}^m$ by $F(x, y) = (x, f(x, y))$. Then $det(dF(a, b)) = det(M) \neq 0$. By inverse function theorem there is an open set $W \subset \mathbb{R}^n \times \mathbb{R}^m$ containing $F(a, b) = (a, 0)$ and an open set in $\mathbb{R}^n \times \mathbb{R}^m$ containing (a, b), which we may take to be of the form $A \times B$, such that $F : A \times B \longrightarrow W$ has a differentiable inverse $h : W \longrightarrow A \times B$. Clearly h is the form $h(x, y) = (x, k(x, y))$ for some differentiable function k (since f is of this form)Let $\pi : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ be defined by $\pi(x, y) = y$; then $\pi \circ F = f$. Therefore $f(x, k(x, y)) = f \circ h(x, y) = (\pi \circ F) \circ h(x, y) = \pi(x, y) = y$ Thus $f(x, k(x, 0)) = 0$ in other words we can define $g(x) = k(x, 0)$

As one might expect the position of the m columns that form M is immaterial. The same proof will work for any $f'(a, b)$ provided that the rank of the matrix is m.

Example $f : \mathbb{R}^2 \longrightarrow \mathbb{R}, f(x,y) = x^2 + y^2 - 1$. $Df = (2x2y)$ Let $(a,b) = (3/5, 4/5)$ M will be $(8/5)$. Now implicit function theorem guarantees the existence and teh uniqueness of g and open intervals $I, J \subset \mathbb{R}, 3/5 \in I, 4/5inJ$ so that $g : I \longrightarrow J$ is differentiable and $x^2 + g(x)^2 - 1 = 0$. One can easily verify this by choosing $I = (-1,1), J = (0,1)$ and $g(x) = \sqrt{1-x^2}$. Note that the uniqueness of $g(x)$ would fail to be true if we did not choose J appropriately.

example Let A be an $m \times (m+n)$ matrix. Consider the function $f : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^m, f(x) = Ax$ Assume that last m columns $C_{n+1}, C_{n+2}, ..., C_{m+n}$ are linearly independent. Break A into blocks $A = [A'|M]$ so that M is the $m \times m$ matrix formed by the last m columns of A. Now the equation $AX = 0$ is a system of m linear equations in $m+n$ unknowns so it has a nontrivial solution. Moreover it can be solved as follows: Let $X = [X_1|X_2]$ where $X_1 \in \mathbb{R}^{n\times 1}$ and $X_2 \in \mathbb{R}^{m \times 1}$ $AX = 0$ implies $A'X_1 + MX_2 = 0 \Rightarrow X_2 = M^{-1}A'X_1$. Now treat f as a function mapping $\mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ by setting $f(X_1, X_2) = AX$. Let $f(a,b) = 0$. Implicit function theorem asserts that there exist open sets $I \subset \mathbb{R}^n, J \subset \mathbb{R}^m$ and a function $g : I \longrightarrow J$ so that $f(x, g(x)) = 0$. By what we did above $g = M^{-1}A'$ is the desired function. So the theorem is true for linear transformations and actually I and J can be chosen $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively.

4.2 Parametric Surfaces

(Following the notation of *Osserman* E^n denotes the Euclidean n-space.) Let D be a domain in the u-plane, $u = (u_1, u_2)$. A parametric surface is simply the image of some differentiable transformation $u : D \longrightarrow E^n$.(A non-empty open set in $\mathbb{R}^2$ is called a domain.)
Let us denote the Jacobian matrix of the mapping $x(u)$ by

$$M = (m_{ij}); m_{ij} = \frac{\partial x_i}{\partial u_j}, i = 1, 2, .., n; j = 1, 2.$$

We introduce the exterior product

$$v \wedge w; w \wedge v \in E^{n(n-1)/2}$$

where the components of $v \wedge w$ are the determinants $\det \begin{pmatrix} v_i & v_j \\ u_i & u_j \end{pmatrix}$ arranged in some fixed order. Finally let

$$G = (g_{ij}) = M^T M; g_{ij} = \frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j}$$

Note that G is a 2×2 matrix. To compute $det(G)$ we recall Lagrange's identity:

$$\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \left(\sum_{k=1}^{n} a_k b_k\right)^2 = \sum_{1 \le i,j \le n} (a_i b_j - a_j b_i)^2$$

Proof of Lagrange's identity is left as an exercise. Using Langrange's identity one can deduce

$$det(G) = \left|\frac{\partial x}{\partial u_1} \wedge x u_2\right|^2 = \sum_{1 \le i,j \le n} \left(\frac{\partial(x_i, x_j)}{\partial(u_1, u_2)}\right)^2$$