Chapter 5

First Fundamental Form

5.1 Tangent Planes

One important tool for studying surfaces is the *tangent plane*. Given a given regular parametrized surface S embedded in $\mathbb{R}^n$ and a point $p \in S$, a *tangent vector* to S at p is a vector in $\mathbb{R}^n$ that is the tangent vector $\alpha'(0)$ of a differential parametrized curve $\alpha\colon (-\epsilon, \epsilon) \to S$ with $\alpha(0) = p$. Then the tangent plane $T_p(S)$ to S at p is the set of all tangent vectors to S at p. This is a set of $\mathbb{R}^3$-vectors that end up being a plane.

An equivalent way of thinking of the tangent plane is that it is the image of $\mathbb{R}^2$ under the linear transformation $Dx(q)$, where x is the map from a domain $D \to S$ that defines the surface, and q is the point of the domain that is mapped onto p. Why is this equivalent? We can show that x is invertible. So given any tangent vector $\alpha'(0)$, we can look at $\gamma = x^{-1} \circ \alpha$, which is a curve in D. Then $\alpha'(0) = (x \circ \gamma)'(0) = (Dx(\gamma(0)) \circ \gamma')(0) = Dx(q)(\gamma'(0))$. Now, γ can be chosen so that $\gamma'(0)$ is *any* vector in $\mathbb{R}^2$. So the tangent plane is the image of $\mathbb{R}^2$ under the linear transformation $Dx(q)$.

Certainly, though, the image of $\mathbb{R}^2$ under an invertible linear transformation (it's invertible since the surface is regular) is going to be a plane including the origin, which is what we'd want a tangent plane to be. (When

I say that the tangent plane includes the origin, I mean that the plane itself consists of all the vectors of a plane through the origin, even though usually you'd draw it with all the vectors emanating from p instead of the origin.)

This way of thinking about the tangent plane is like considering it as a "linearization" of the surface, in the same way that a tangent line to a function from $\mathbb{R} \to \mathbb{R}$ is a linear function that is locally similar to the function. Then we can understand why $Dx(q)(\mathbb{R}^2)$ makes sense: in the same way we can "replace" a function with its tangent line which is the image of $\mathbb{R}$ under the map $t \mapsto f'(p)t + C$, we can replace our surface with the image of $\mathbb{R}^2$ under the map $Dx(q)$.

The interesting part of seeing the tangent plane this way is that you can then consider it as having a basis consisting of the images of $(1, 0)$ and $(0, 1)$ under the map $Dx(q)$. These images are actually just (if the domain in $\mathbb{R}^2$ uses u_1 and u_2 as variables) $\frac{\partial x}{\partial u_1}$ and $\frac{\partial x}{\partial u_2}$ (which are n-vectors).

5.2 The First Fundamental Form

Nizam mentioned the First Fundamental Form. Basically, the FFF is a way of finding the length of a tangent vector (in a tangent plane). If w is a tangent vector, then $|w|^2 = w \cdot w$. Why is this interesting? Well, it becomes more interesting if you're considering w not just as its $\mathbb{R}^3$ coordinates, but as a linear combination of the two basis vectors $\frac{\partial x}{\partial u_1}$ and $\frac{\partial x}{\partial u_2}$. Say $w = a\frac{\partial x}{\partial u_1} + b\frac{\partial x}{\partial u_2}$; then

$$\begin{aligned} |w|^2 &= \left(a\frac{\partial x}{\partial u_1} + b\frac{\partial x}{\partial u_2}\right) \cdot \left(a\frac{\partial x}{\partial u_1} + b\frac{\partial x}{\partial u_2}\right) \\ &= a^2\frac{\partial x}{\partial u_1} \cdot \frac{\partial x}{\partial u_1} + 2ab\frac{\partial x}{\partial u_1} \cdot \frac{\partial x}{\partial u_2} + b^2\frac{\partial x}{\partial u_2} \cdot \frac{\partial x}{\partial u_2}. \end{aligned} \tag{5.1}$$

Let's deal with notational differences between do Carmo and Osserman. do Carmo writes this as $Ea^2 + 2Fab + Gb^2$, and refers to the whole thing as $I_p \colon T_p(S) \to \mathbb{R}$.[1] Osserman lets $g_{11} = E$, $g_{12} = g_{21} = F$ (though he never

[1] Well, actully he's using u' and v' instead of a and b at this point, which is because these coordinates come from a tangent vector, which is to say they are the $u'(q)$ and $v'(q)$

makes it too clear that these two are equal), and $g_{22} = G$, and then lets the matrix that these make up be G, which he also uses to refer to the whole form. I am using Osserman's notation.

Now we'll calculate the FFF on the cylinder over the unit circle; the parametrized surface here is $x\colon (0, 2\pi) \times \mathbb{R} \to S \subset \mathbb{R}^3$ defined by $x(u, v) = (\cos u, \sin u, v)$. (Yes, this misses a vertical line of the cylinder; we'll fix this once we get away from *parametrized* surfaces.) First we find that $\frac{\partial x}{\partial u} = (-\sin u, \cos u, 0)$ and $\frac{\partial x}{\partial v} = (0, 0, 1)$. Thus $g_{11} = \frac{\partial x}{\partial u} \cdot \frac{\partial x}{\partial u} = \sin^2 u + \cos^2 u = 1$, $g_{21} = g_{12} = 0$, and $g_{22} = 1$. So then $|w|^2 = a^2 + b^2$, which basically means that the length of a vector in the tangent plane to the cylinder is the same as it is in the $(0, 2\pi) \times \mathbb{R}$ that it's coming from.

As an exercise, calculate the first fundamental form for the sphere S^2 parametrized by $x\colon (0, \pi) \times (0, 2\pi) \to S^2$ with

$$x(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta). \tag{5.2}$$

We first calculate that $\frac{\partial x}{\partial \theta} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)$ and $\frac{\partial x}{\partial \varphi} = (-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0)$. So we find eventually that $|w|^2 = a^2 + b^2\sin^2\theta$. This makes sense — movement in the φ direction (latitudinally) should be "worth more" closer to the equator, which is where $\sin^2\theta$ is maximal.

5.3 Area

If we recall the exterior product from last time, we can see that $\left|\frac{\partial x}{\partial u} \wedge \frac{\partial x}{\partial v}\right|$ is the area of the parallelogram determined by $\frac{\partial x}{\partial u}$ and $\frac{\partial x}{\partial v}$. This is analogous to the fact that in 18.02 the magnitude of the cross product of two vectors is the area of the parallelogram they determine. Then $\int_Q \left|\frac{\partial x}{\partial u} \wedge \frac{\partial x}{\partial v}\right| du dv$ is the area of the bounded region Q in the surface. But Nizam showed yesterday

of some curve in the domain D.

that Lagrange's Identity implies that

$$\left|\frac{\partial x}{\partial u} \wedge \frac{\partial x}{\partial v}\right|^2 = \left|\frac{\partial x}{\partial u}\right|^2 \left|\frac{\partial x}{\partial v}\right|^2 - \left(\frac{\partial x}{\partial u} \cdot \frac{\partial x}{\partial v}\right)^2 \tag{5.3}$$

Thus $\left|\frac{\partial x}{\partial u} \wedge \frac{\partial x}{\partial v}\right| = \sqrt{g_{11}g_{22} - g_{12}^2}$. Thus, the area of a bounded region Q in the surface is $\int_Q \sqrt{g_{11}g_{22} - g_{12}^2}\mathrm{d}u\mathrm{d}v$.

For example, let us compute the surface area of a torus; let's let the radius of a meridian be r and the longitudinal radius be a. Then the torus (minus some tiny strip) is the image of $x\colon (0, 2\pi) \times (0, 2\pi) \to S^1 \times S^1$ where $x(u, v) = ((a + r\cos u)\cos v, (a + r\cos u)\sin v), r\sin u)$. Then $\frac{\partial x}{\partial u} = (-r\sin u\cos v, -r\sin u\sin v, r\cos u)$, and $\frac{\partial x}{\partial v} = (-(a+r\cos u)\sin v, (a+r\cos u)\cos v, 0)$. So $g_{11} = r^2$, $g_{12} = 0$, and $g_{22} = (r\cos u + a)^2$. Then $\sqrt{g_{11}g_{22} - g_{12}^2} = r(r\cos u + a)$. Integrating this over the whole square, we get

$$\begin{aligned} A &= \int_0^{2\pi}\int_0^{2\pi} (r^2\cos u + ra)\mathrm{d}u\mathrm{d}v \\ &= \left(\int_0^{2\pi} (r^2\cos u + ra)\mathrm{d}u\right)\left(\int_0^{2\pi} \mathrm{d}v\right) \\ &= (r^2\sin 2\pi + ra2\pi)(2\pi) = 4\pi^2 ra \end{aligned}$$

And this is the surface area of a torus!

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