Chapter 7

Tangent Planes

Reading: Do Carmo sections 2.4 and 3.2

Today I am discussing

1. Differentials of maps between surfaces
2. Geometry of Gauss map

7.1 Tangent Planes; Differentials of Maps Between Surfaces

7.1.1 Tangent Planes

Recall from previous lectures the definition of *tangent plane.*

(Proposition 2-4-1). *Let* $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ *be a parameterization of a regular surface* S *and let* $q \in U$. *The vector subspace of dimension 2,*

$$d\mathbf{x}_q(\mathbb{R}^2) \subset \mathbb{R}^3 \tag{7.1}$$

coincides with the set of tangent vectors to S *at* $\mathbf{x}(q)$. *We call the plane* $d\mathbf{x}_q(\mathbb{R}^2)$ *the* **Tangent Plane** *to* S *at* p, *denoted by* $T_p(S)$.

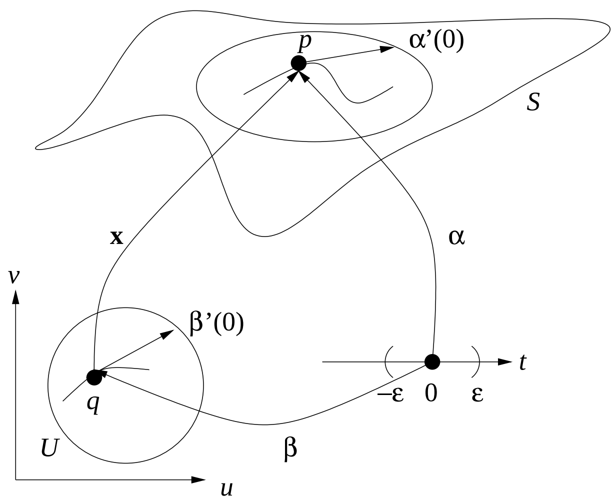

Figure 7.1: Graphical representation of the map dx_q that sends $\beta'(0) \in T_q(\mathbb{R}^2$ to $\alpha'(0) \in T_p(S)$.

Note that the plane $dx_q(\mathbb{R}^2)$ does not depend on the parameterization $\mathbf{x}$. However, the choice of the parameterization determines the basis on $T_p(S)$, namely $\{(\frac{\partial \mathbf{x}}{\partial u})(q), (\frac{\partial \mathbf{x}}{\partial v})(q)\}$, or $\{\mathbf{x}_u(q), \mathbf{x}_v(q)\}$.

7.1.2 Coordinates of $w \in T_p(S)$ in the Basis Associated to Parameterization $\mathbf{x}$

Let w be the velocity vector $\alpha'(0)$, where $\alpha = \mathbf{x} \circ \beta$ is a curve in the surface S, and the map $\beta : (-\epsilon, \epsilon) \to U$, $\beta(t) = (u(t), v(t))$. Then in the basis of $\{\mathbf{x}_u(q), \mathbf{x}_v(q)\}$, we have

$$w = (u'(0), v'(0)) \tag{7.2}$$

7.1.3 Differential of a (Differentiable) Map Between Surfaces

It is natural to extend the idea of differential map from $T(\mathbb{R}^2) \to T(S)$ to $T(S_1) \to T(S_2)$.

Let S_1, S_2 be two regular surfaces, and a differential mapping $\varphi \subset S_1 \to S_2$ where V is open. Let $p \in V$, then all the vectors $w \in T_p(S_1)$ are velocity vectors $\alpha'(0)$ of some differentiable parameterized curve $\alpha : (-\epsilon, \epsilon) \to V$ with $\alpha(0) = p$.

Define $\beta = \varphi \circ \alpha$ with $\beta(0) = \varphi(p)$, then $\beta'(0)$ is a vector of $T_{\varphi(p)}(S_2)$.

(Proposition 2-4-2). *Given w, the velocity vector $\beta'(0)$ does not depend on the choice of α. Moreover, the map*

$$d\varphi_p : T_p(S_1) \to T_{\varphi(p)}(S_2) \tag{7.3}$$

$$d\varphi_p(w) = \beta'(0) \tag{7.4}$$

is linear. We call the linear map $d\varphi_p$ to be the **differential** *of φ at $p \in S_1$.*

Proof. Suppose φ is expressed in $\varphi(u, v) = (\varphi_1(u, v), \varphi_2(u, v))$, and $\alpha(t) = (u(t), v(t)), t \in (-\epsilon, \epsilon)$ is a regular curve on the surface S_1. Then

$$\beta(t) = (\varphi_1(u(t), v(t)), \varphi_2(u(t), v(t)). \tag{7.5}$$

Differentiating β w.r.t. t, we have

$$\beta'(0) = \left(\frac{\partial \varphi_1}{\partial u} u'(0) + \frac{\varphi_1}{\partial v} v'(0), \frac{\partial \varphi_2}{\partial u} u'(0) + \frac{\varphi_2}{\partial v} v'(0) \right) \tag{7.6}$$

in the basis of $(\bar{\mathbf{x}}_u, \bar{\mathbf{x}}_v)$.

As shown above, $\beta'(0)$ depends on the map φ and the coordinates of $(u'(0), v'(0)$ in the basis of $\{\mathbf{x}_u, \mathbf{x}_v\}$. Therefore, it is independent on the choice of α.

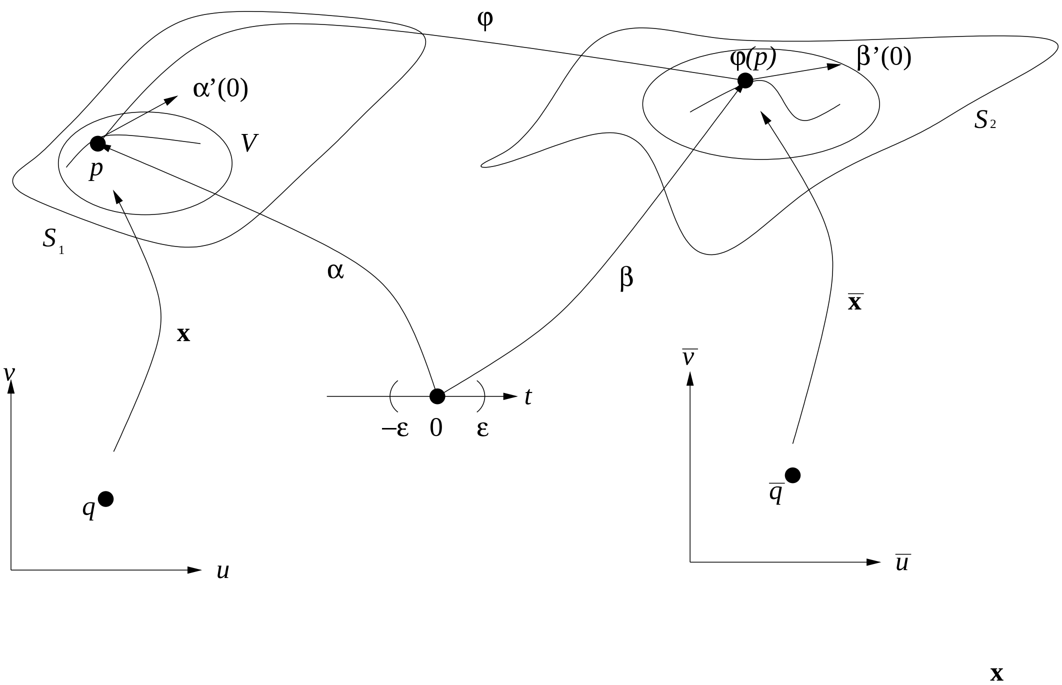

Figure 7.2: Graphical representation of the map $d\varphi_p$ that sends $\alpha'(0) \in T_q(S_1)$ to $\beta'(0) \in T_p(S_2)$.

Moreover, Equation 7.6 can be expressed as

$$\beta'(0) = d\varphi_p(w) = \begin{pmatrix} \frac{\partial \varphi_1}{\partial u} & \frac{\partial \varphi_1}{\partial v} \\ \frac{\partial \varphi_2}{\partial u} & \frac{\partial \varphi_2}{\partial v} \end{pmatrix} \begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix} \tag{7.7}$$

which shows that the map $d\varphi_p$ is a mapping from $T_p(S_1)$ to $T_{\varphi(p)}(S_2)$. Note that the 2×2 matrix is respect to the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ of $T_p(S_1)$ and $\{\bar{\mathbf{x}}_u, \bar{\mathbf{x}}_v\}$ of $T_{\varphi(p)}(S_2)$ respectively. $\square$

We can define the differential of a (differentiable) function $f : U \subset S \to R$ at $p \in U$ as a linear map $df_p : T_p(S) \to \mathbb{R}$.

Example 2-4-1: Differential of the height function Let $v \in \mathbb{R}^3$. Consider the map

$$h : S \subset \mathbb{R}^3 \to \mathbb{R} \tag{7.8}$$

$$h(p) = v \cdot p, p \in S \tag{7.9}$$

We want to compute the differential $dh_p(w), w \in T_p(S)$. We can choose a differential curve $\alpha : (-\epsilon, \epsilon)) \to S$ such that $\alpha(0) = p$ and $\alpha'(0) = w$. We are able to choose such α since the differential $dh_p(w)$ is independent on the choice of α. Thus

$$h(\alpha(t)) = \alpha(t) \cdot v \tag{7.10}$$

Taking derivatives, we have

$$dh_p(w) = \frac{d}{dt} h(\alpha(t))|_{t=0} = \alpha'(0) \cdot v = w \cdot v \tag{7.11}$$

Example 2-4-2: Differential of the rotation Let $S^2 \subset \mathbb{R}^3$ be the unit sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\} \tag{7.12}$$

Consider the map

$$R_{z,\theta} : \mathbb{R}^3 \to \mathbb{R}^3 \tag{7.13}$$

be the rotation of angle θ about the z axis. When $R_{z,\theta}$ is restricted to S^2, it becomes a differential map that maps S^2 into itself. For simplicity, we denote the restriction map $R_{z,\theta}$. We want to compute the differential $(dR_{z,\theta})_p(w), p \in S^2, w \in T_p(S^2)$. Let $\alpha : (-\epsilon, \epsilon) \to S^2$ be a curve on S^2 such that $\alpha(0) = p, \alpha'(0) = w$. Now

$$(dR_{z,\theta})(w) = \frac{d}{dt}(R_{z,\theta} \circ \alpha(t))_{t=0} = R_{z,\theta}(\alpha'(0)) = R_{z,\theta}(w) \tag{7.14}$$

7.1.4 Inverse Function Theorem

All we have done is extending differential calculus in $\mathbb{R}^2$ to regular surfaces. Thus, it is natural to have the Inverse Function Theorem extended to the regular surfaces.

A mapping $\varphi : U \subset S_1 \to S_2$ is a **local diffeomorphism** at $p \in U$ if there exists a neighborhood $V \subset U$ of p, such that φ restricted to V is a diffeomorphism onto the open set $\varphi(V) \subset S_2$.

(Proposition 2-4-3). *Let S_1, S_2 be regular surfaces and $\varphi : U \subset S_1 \to S_2$ a differentiable mapping. If $d\varphi_p : T_p(S_1) \to T_{\varphi(p)}(S_2)$ at $p \in U$ is an isomorphism, then φ is a local diffeomorphism at p.*

The proof is a direct application of the inverse function theorem in $\mathbb{R}^2$.

7.2 The Geometry of Gauss Map

In this section we will extend the idea of curvature in curves to regular surfaces. Thus, we want to study how rapidly a surface S pulls away from the tangent plane $T_p(S)$ in a neighborhood of $p \in S$. This is equivalent to measuring the rate of change of a unit normal vector field N on a neighborhood of p. We will show that this rate of change is a linear map on $T_p(S)$ which is self adjoint.

7.2.1 Orientation of Surfaces

Given a parameterization $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ of a regular surface S at a point $p \in S$, we choose a unit normal vector at each point $\mathbf{x}(U)$ by

$$N(q) = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}(q), q \in \mathbf{x}(U) \tag{7.15}$$

We can think of N to be a map $N : \mathbf{x}(U) \to \mathbb{R}^3$. Thus, each point $q \in \mathbf{x}(U)$ has a normal vector associated to it. We say that N is a **differential field**

of unit normal vectors on U.

We say that a regular surface is **orientable** if it has a differentiable field of unit normal vectors defined on the whole surface. The choice of such a field N is called an **orientation** of S. An example of non-orientable surface is Möbius strip (see Figure 3).

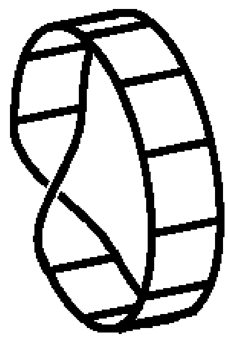

Figure 7.3: Möbius strip, an example of non-orientable surface.

In this section (and probably for the rest of the course), we will only study regular orientable surface. We will denote S to be such a surface with an orientation N which has been chosen.

7.2.2 Gauss Map

(Definition 3-2-1). *Let* $S \subset \mathbb{R}^3$ *be a surface with an orientation* N *and* $S^2 \subset \mathbb{R}^3$ *be the unit sphere*

$$S^2 = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\}. \tag{7.16}$$

The map $N : S \to S^2$ *is called the* **Gauss map**.

The map N is differentiable since the differential,

$$dN_p : T_p(S) \to T_{N(p)}(S^2) \tag{7.17}$$

at $p \in S$ is a linear map.

For a point $p \in S$, we look at each curve $\alpha(t)$ with $\alpha(0) = p$ and compute $N \circ \alpha(t) = N(t)$ where we define that map $N : (-\epsilon, \epsilon) \to S^2$ with the same notation as the normal field. By this method, we restrict the normal vector N to the curve $\alpha(t)$. The tangent vector $N'(0) \in T_p(S^2)$ thus measures the rate of change of the normal vector N restrict to the curve $\alpha(t)$ at $t = 0$. In other words, dN_p measure how N pulls away from $N(p)$ in a neighborhood of p. In the case of the surfaces, this measure is given by a linear map.

Example 3-2-1 (Trivial) Consider S to be the plane $ax + by + cz + d = 0$, the tangent vector at any point $p \in S$ is given by

$$N = \frac{(a, b, c)}{\sqrt{a^2 + b^2 + c^2}} \tag{7.18}$$

Since N is a constant throughout S, $dN = 0$.

Example 3-2-2 (Gauss map on the Unit Sphere)

Consider $S = S^2 \subset \mathbb{R}^3$, the unit sphere in the space $\mathbb{R}^3$. Let $\alpha(t) = (x(t), y(t), z(t))$ be a curve on S, then we have

$$2xx' + 2yy' + 2zz' = 0 \tag{7.19}$$

which means that the vector (x, y, z) is normal to the surface at the point (x,y,z). We will choose $N = (-x, -y, -z)$ to be the normal field of S. Restricting to the curve $\alpha(t)$, we have

$$N(t) = (-x(t), -y(t), -z(t)) \tag{7.20}$$

and therefore

$$dN(x'(t), y'(t), z'(t)) = (-x'(t), -y'(t), -z'(t)) \tag{7.21}$$

or $dN_p(v) = -v$ for all $p \in S$ and $v \in T_p(S^2)$.

Example 3-2-4 (Exercise: Gauss map on a hyperbolic paraboloid) Find the differential $dN_{p=(0,0,0)}$ of the normal field of the paraboloid $S \subset \mathbb{R}^3$ defined by

$$\mathbf{x}(u,v) = (u, v, v^2 - u^2) \tag{7.22}$$

under the parameterization $\mathbf{x} : U \subset \mathbb{R}^2 \to S$.

7.2.3 Self-Adjoint Linear Maps and Quadratic Forms

Let V now be a vector space of dimension 2 endowed with an inner product $\langle\ ,\ \rangle$.

Let $A : V \to V$ be a linear map. If $\langle Av, w\rangle = \langle v, Aw\rangle$ for all $v, w \in V$, then we call A to be a **self-adjoint** linear map.

Let $\{e_1, e_2\}$ be a orthonormal basis for V and $(\alpha_{ij}), i, j = 1, 2$ be the matrix elements of A in this basis. Then, according to the axiom of self-adjoint, we have

$$\langle Ae_i, e_j\rangle = \alpha_{ij} = \langle e_i, Ae_j\rangle = \langle Ae_j, e_i\rangle = \alpha_{ji} \tag{7.23}$$

There A is **symmetric**.

To each self-adjoint linear map, there is a **bilinear map** $B : V \times V \to \mathbb{R}$ given by

$$B(v,w) = \langle Av, w\rangle \tag{7.24}$$

It is easy to prove that B is a bilinear symmetric form in V.

For each bilinear form B in V, there is a **quadratic form** $Q : V \to \mathbb{R}$ given by

$$Q(v) = B(v,v), v \in V. \tag{7.25}$$

Exercise (Trivial): Show that

$$B(u,v) = \frac{1}{2}\left[Q(u+v) - Q(v) - Q(u)\right] \tag{7.26}$$

Therefore, there is a 1-1 correspondence between quadratic form and self-adjoint linear maps of V.

Goal for the rest of this section: Show that given a self-adjoint linear map $A : V \to V$, there exist a orthonormal basis for V such that, relative to this basis, the matrix A is diagonal matrix. Moreover, the elements of the diagonal are the maximum and minimum of the corresponding quadratic form restricted to the unit circle of V.

(Lemma (Exercise)). *If* $Q(x,y) = ax^2 = 2bxy + cy^2$ *restricted to* $\{(x,y); x^2 + y^2 = 1\}$ *has a maximum at* $(1,0)$*, then* $b = 0$

Hint: Reparametrize (x,y) using $x = \cos t, y = \cos t, t \in (-\epsilon, 2\pi + \epsilon)$ and set $\frac{dQ}{dt}|_{t=0} = 0$.

(Proposition 3A-1). *Given a quadratic form* Q *in* V*, there exists an orthonormal basis* $\{e_1, e_2\}$ *of* V *such that if* $v \in V$ *is given by* $v = xe_1 + ye_2$*, then*

$$Q(v) = \lambda_1 x^2 + \lambda_2 y^2 \tag{7.27}$$

where $\lambda_i, i = 1, 2$ *are the maximum and minimum of the map* Q *on* $|v| = 1$ *respectively.*

Proof. Let λ_1 be the maximum of Q on the circle $|v| = 1$, and e_1 to be the unit vector with $Q(e_1) = \lambda_1$. Such e_1 exists by continuity of Q on the compact set $|v| = 1$.

Now let e_2 to be the unit vector orthonormal to e_1, and let $\lambda_2 = Q(e_2)$. We will show that this set of basis satisfy the proposition.

Let B be a bilinear form associated to Q. If $v = xe_1 + ye_2$, then

$$Q(v) = B(v,v) = \lambda_1 x^2 + 2bxy + \lambda_2 y^2 \tag{7.28}$$

where $b = B(e_1, e_2)$. From previous lemma, we know that $b = 0$. So now it suffices to show that λ_2 is the minimum of Q on $|v| = 1$. This is trivial since we know that $x^2 + y^2 = 1$ and

$$Q(v) = \lambda_1 x^2 + \lambda_2 y^2 \geq \lambda_2(x^2 + y^2) = \lambda_2 \tag{7.29}$$

as $\lambda 2 \leq \lambda 1$. $\square$

If $v \neq 0$, then v is called the **eigenvector** of $A : V \to V$ if $Av = \lambda v$ for some real λ. We call the λ the corresponding **eigenvalue**.

(Theorem 3A-1). *Let $A : V \to V$ be a self-adjoint linear map, then there exist an orthonormal basis $\{e_1, e_2\}$ of V such that*

$$A(e_1) = \lambda_1 e_1, \quad A(e_2) = \lambda_2 e_2. \tag{7.30}$$

Thus, A is diagonal in this basis and $\lambda_i, i = 1, 2$ are the maximum and minimum of $Q(v) = \langle Av, v\rangle$ on the unit circle of V.

Proof. Consider $Q(v) = \langle Av, v\rangle$ where $v = (x, y)$ in the basis of $e_i, i = 1, 2$. Recall from the previous lemma that $Q(x, y) = ax^2 + cy^2$ for some $a, c \in \mathbb{R}$. We have $Q(e_1) = Q(1, 0) = a, Q(e_2) = Q(0, 1) = c$, therefore $Q(e_1 + e_2) = Q(1, 1) = a + c$ and

$$B(e_1, e_2) = \frac{1}{2}[Q(e_1 + e_2) - Q(e_1) - Q(e_2)] = 0 \tag{7.31}$$

Thus, Ae_1 is either parallel to e_1 or equal to 0. In any case, we have $Ae_1 = \lambda_1 e_1$. Using $B(e_1, e_2) = \langle Ae_2, e_1\rangle = 0$ and $\langle Ae_2, e_2\rangle = \lambda_2$, we have $Ae_2 = \lambda_2 e_2$. $\square$

Now let us go back to the discussion of Gauss map.

(Proposition 3-2-1). *The differential map $dN_p : T_p(S) \to T_p(S)$ of the Gauss map is a self-adjoint linear map.*

Proof. It suffices to show that

$$\langle dN_p(w_1), w_2\rangle = \langle w_1, dN_p(w_2)\rangle \tag{7.32}$$

for the basis $\{w_1, w_2\}$ of $T_p(S)$.

Let $\mathbf{x}(u, v)$ be a parameterization of S at p, then $\mathbf{x}_u, \mathbf{x}_v$ is a basis of $T_p(S)$. Let $\alpha(t) = \mathbf{x}(u(t), v(t))$ be a parameterized curve in S with $\alpha(0) = p$, we have

$$dN_p(\alpha'(0)) = dN_p(x_u u'(0) + x_v v'(0)) \tag{7.33}$$

$$= \frac{d}{dt} N(u(t), v(t))|_{t=0} \tag{7.34}$$

$$= N_u u'(0) + N_v v'(0) \tag{7.35}$$

with $dN_p(\mathbf{x}_u) = N_u$ and $dN_p(\mathbf{x}_v) = N_v$. So now it suffices to show that

$$\langle N_u, \mathbf{x}_v \rangle = \langle \mathbf{x}_u, N_v \rangle \tag{7.36}$$

If we take the derivative of $\langle N, \mathbf{x}_u \rangle = 0$ and $\langle N, \mathbf{x}_v \rangle = 0$, we have

$$\langle N_v, \mathbf{x}_u \rangle + \langle N, \mathbf{x}_u v \rangle = 0 \tag{7.37}$$

$$\langle N_u, \mathbf{x}_v \rangle + \langle N, \mathbf{x}_v u \rangle = 0 \tag{7.38}$$

Therefore

$$\langle N_u, \mathbf{x}_v \rangle = - \langle N, \mathbf{x}_u v \rangle = \langle N_v, \mathbf{x}_u \rangle \tag{7.39}$$

□