$\alpha = 0$: $SU(ab/A) \geq SU(a/A) \geq SU(a/Ab)$.
↑ exercise.

α limit: By ind hyp. (since α on right-hand-side).

$SU(b/A) \geq \alpha+1$ so $\exists c$ s.t. $SU(b/Ac) \geq \alpha$, $b \not\downarrow_A c$.

WMA $c \downarrow_{Ab} a$.

$\Rightarrow$ ① $SU(a/Ab) = SU(a/Abc)$
By ind hyp, $SU(ab/Ac) \geq SU(a/Abc) + \alpha = SU(a/Ab) + \alpha$.

② $ab \not\downarrow_A c$ ~~otherwise $a \downarrow bc$~~ since $b \not\downarrow_A c$

$\Rightarrow$ $SU(ab/A) \geq SU(ab/Ac) + 1 \geq SU(a/Ab) + \alpha + 1$.

Now other inequality, we prove by induction:

$\forall \alpha$ if $SU(ab/A) \geq \alpha \Rightarrow SU(a/Ab) \oplus SU(b/A) \geq \alpha$.

Again $\alpha = 0$, limit ✓.

Continued on next page.

Assume $SU(ab/A) \geq \alpha+1$. Then $\exists c$ s.t. $ab \not\downarrow_A c$ & $SU(ab/Ac) \geq \alpha$.

~~Two cases~~. $\Rightarrow SU(a/Abc) \oplus SU(b/Ac) \geq \alpha$ by ind

Either $b \not\downarrow_A c$ or $a \not\downarrow_{Ab} c$ (otherwise by trans we get a

In either case $SU(a/Ab) \oplus SU(b/A) \geq SU(a/Abc) \oplus SU(b/Ac) \geq \alpha+1$

II. If $a \downarrow_A b \Rightarrow SU(ab/A) = SU(a/A) \oplus SU(b/A)$
($SU(a/A) = SU(a/Ab)$)

Assume $SU(a/A) \oplus SU(b/A) \geq \alpha+1$.

So wlog assume $\exists \beta, \gamma$ s.t. $SU(a/A) \geq \beta+1$, & $SU(b/A) \geq \gamma$ & $\beta \oplus \gamma \geq \alpha$. } requires ordinal

$\exists c$ $a \not\downarrow_A c$ $SU(a/Ac) \geq \beta$.

WMA $c \downarrow_{Aa} b \Rightarrow ac \downarrow_A b \Rightarrow a \downarrow_{Ac} b$

$\Rightarrow SU(ab/Ac) \geq \beta \oplus \gamma$.

Explicitly $SU(ab/Ac) \geq SU(a/Ac) \oplus SU(b/Ac) \geq$

$\Rightarrow SU(ab/A) \geq \beta \oplus \gamma + 1 \geq \alpha+1$. So we have equ writing out ordinal remarks:

Want $\beta \oplus \gamma \geq \alpha+1 \Rightarrow \exists \beta' \ \beta \geq \beta'+1 \ \ \beta' \oplus \gamma \geq \alpha$ or $\gamma \ldots$

write $\beta = \sum \omega^{\alpha_i} n_i \qquad \gamma = \sum \omega^{\alpha_i} m_i \qquad \beta \oplus \gamma = \sum \omega^{\alpha_i}(m_i+n_i) \geq \alpha+1$

write $\alpha = \sum \omega^{\alpha_i} k_i$

Fact: ~~$\alpha \oplus \beta$~~ $(\alpha, \beta) \mapsto \alpha \oplus \beta$ is minimal s.t.

if $(\alpha+1) \oplus \beta \geq \alpha \oplus \beta + 1$ and symmetry.

$\beta \oplus \gamma = \sum \omega^{\alpha_i}(m_i + n_i) \geq \left(\sum \omega^{\alpha_i} k_i\right) + 1.$

Let j be least s.t. $k_j < m_j + n_j$

If $n_j = 0 \quad k_j < m_j.$

Let $\gamma' = \sum_{i<j} \omega^{\alpha_i} m_i + \sum_{i \geq j} \omega^{\alpha_i} k_i$

so $\gamma \geq \gamma'+1 \quad$ & $\gamma' \oplus \beta \geq \alpha.$

Otherwise define $\beta' = \sum_{i<j} \omega^{\alpha_i} n_i + \omega^{\alpha_j}(n_j - 1) + \sum_{i > j} \omega^{\alpha_i} k_i$

so $\beta \geq \beta'+1 \quad$ & $\beta' \oplus \gamma \geq \alpha.$

One more inequality (whose proof is the messiest).

III ("Higher exponent symmetry").

Assume $SU(a/A) \geq SU(a/Ab) + \omega^\alpha \cdot n$.
(ie very dependent on b).

Then $SU(b/A) \geq SU(b/Aa) + \omega^\alpha \cdot n$.

This property is useful.

Small claim: Assume $SU(a/B) = \omega^\alpha$ and $B \subseteq C$ & $a \underset{B}{\perp} C$ & $b \underset{B}{\equiv} a$ s.t. $b \underset{B}{\not\perp} C$ $\Rightarrow a \underset{C}{\perp} b$.

Proof Assume $a \underset{C}{\not\perp} b$. Then $\omega^\alpha = SU(a/C) \leq SU(ab/C)$

$\leq \underbrace{SU(a/bC)}_{<\omega^\alpha} \oplus \underbrace{SU(b/C)}_{<\omega^\alpha} < \omega^\alpha$. □.

4/7. Defn ① For every two contradicting formulas $\varphi(x,y)$, $\psi(x,y)$ (could be tuples)

define $R(p(x), \varphi, \psi, 2)$ $\in \omega \cup \{\infty\}$ inductively as follows:

- $R(p, \varphi, \psi, 2) \geq 0$ if $p(x)$ is consistent.
- $R(p, \varphi, \psi, 2) \geq n+1$ if $\exists b$ s.t. $R(p(x) \wedge \varphi(x,b), \varphi, \psi, 2) \geq n$

  and $R(p(x) \wedge \psi(x,b), \varphi, \dots) \geq n$.

(2) The pair (φ, ψ) is stable if $R(\overset{T}{x=x}, \varphi, \psi, 2) < \infty$.

(3) φ is stable if (φ, ψ) is stable $\forall \psi$ contradicting φ.

T is stable if all formulas are.

finite tuples since talking about a single formula.

<u>Defn</u> Let $p \in S(A)$, $\varphi(x, y)$ a formula.

A φ-definition for p (over A) is a partial type $d_\varphi p(y)$ $/A$

satisfying:

- $|d_\varphi p| \leq |T|$.
- $\forall b \in A$ (of the length of y), $\varphi(x, b) \in p$ iff $\models d_\varphi p(b)$.

(2) A definition of $p(x)$ is a set $\{d_\varphi p : \varphi(x, y)\}$

such that each $d_\varphi p$ is a φ-def for p.

(3) A good defn for p is a definition $\{d_\varphi p\}$ st.

$\forall B$ the type $q = \{\varphi(x, b) : \varphi(x, y), b \in B \text{ st. } \models d_\varphi(b)\}$

is a complete consistent type.

(4) p is (well) definable if it has a (good) definition.

(a)

Now: If $\varphi(x,b) \in p$ then $\chi(x,c) \wedge \varphi(x,b) \in p$

$\Rightarrow \rho_\psi(b)$ is true.

On the other hand if $\psi(x,b) \in p$ then $\chi(x,c) \wedge \psi(x,b) \in p$

$\Rightarrow \rho_\psi(b)$ is false (o.w. $R(\chi, \varphi, \psi, 2) \geq n+1$)

Let $d_\varphi p(y) = \{ \rho_\psi(y) : \psi$ contradicting $\varphi \}$.

Then $|d_\varphi p| \leq |T|$ & $d_\varphi p(y)$ is over A.

If $\varphi(x,b) \in p$ then $\models d_\varphi p(b)$ from (a).

If $\varphi(x,b) \notin p$ then since p is complete

$\exists \psi$ contradicting φ s.t. $\psi(x,b) \in p$.

So $\not\models \rho_\psi(b) \Rightarrow \not\models d_\varphi p(b)$.

(2) ⇒ (3): count possible definitions.

(3) ⇒ (4): eg take $\lambda = 2^{|T|}$ so $(\lambda + |T|)^{|T|} = \lambda$.

(4) ⇒ (1): Assume ¬(1) and let λ be any cardinal.

let κ be least s.t. $2^\kappa > \lambda$. so $\kappa \leq \lambda$

So $2^{<\kappa} = \sum_{\mu < \kappa} 2^\mu \leq \lambda \cdot \lambda = \lambda$.

So by assumption we have φ, ψ contradictory s.t. $R(\overset{T}{x=x}, \varphi, \psi, 2) = \infty$.

So by compactness we find $\{a_\tau : \tau \in 2^\kappa\}$ and $\{b_\sigma \cdot \sigma \in 2^{<\kappa}\}$ s.t. $\forall \tau \in 2^\kappa, \alpha < \kappa$ we have if $\tau(\alpha) = 0$ then $\varphi(a_\tau, b_{\tau|_\alpha})$

if $\tau(\alpha) = 1$ then $\Psi(a_\tau, b_{\tau|_\alpha})$.

Let $B = \{b_\sigma\}$ then $|B| = 2^{<\kappa} \leq \lambda$

But we found $2^\kappa > \lambda$ contradictory φ-types over B. □

Corollary TFAE:

(1) T stable

(2) ~~[illegible]~~ Every complete type is definable.

(3) $\forall A, \ |S(A)| \leq (|A| + |T|)^{|T|}$

(4) $\exists \lambda$ st. $|A| \leq \lambda \Rightarrow |S(A)| \leq \lambda$.

Sketchy proof (1) ⇒ (2) ✓. ⇒ (3). notice $[(|A|+|T|)^{|T|}]^{|T|} = (|A|+|T|)^{|T|}$.

(3) ⇒ (4). Take $\lambda = 2^{|T|}$ & use proof of (4) ⇒ (1) in theorem. □.

Defn Let $A \subseteq B$, $p \in S(B)$. Then p is non-splitting over A if ~~[illegible]~~ $\forall \varphi(x, y)$ & $b, c \in B$ of length of y, then if $b \equiv_A c$ then $\varphi(x, b) \in p$ iff $\varphi(x, c) \in p$.

In other words, if $a \models p$ and $b \equiv_A c$ $(b, c \in B)$

then $b \equiv_{Aa} c$.

<u>Defn</u> Let $\kappa > |T|$. A set $\mathcal{M} \subseteq \mathcal{U}$ is κ-saturated if

$\forall A \subseteq M$ $|A| < \kappa$, $\forall p \in S(A)$, p is realised in $\mathcal{M}$

<u>Fact</u> $\forall A$ $\exists \mathcal{M} \supseteq A$ s.t. $\mathcal{M}$ is $|T|^+$-saturated.

<u>Lemma</u> Let $\mathcal{M}$ be $|T|^+$saturated, $p \in S(\mathcal{M})$ definable.

Then (i) p has a unique definition (up to equivalence)

(ii) the unique definition is good.

- $\forall B$, let $p|_B$ be the type resulting from the application of the definition to B.

(iii). $\forall B \supseteq \mathcal{M}$, $p|_B$ is a nonsplitting extension of p.

<u>Proof</u> (i) Assume $\{d_\varphi p\}$ and $\{d'_\varphi p\}$ are both definitions & not equivalent.

So $\exists \varphi$ s.t. $d_\varphi p \not\equiv d'_\varphi p$.

ie $\exists b$ (not in $\mathcal{M}$) s.t. say $\models d_\varphi p(b)$ &
$\not\models d'_\varphi p(b)$

~~So there exists~~

So $tp(b/\mathcal{M})$ contradicts over $d_{\varphi}' p(y)$.

$\Rightarrow \exists c \in \mathcal{M}$ & $\chi(y, z)$ s.t. $\models \chi(b, c)$ and

$\chi(y, c)$ contradicts $d_{\varphi}' p(y)$.

Let A = set of parameters used in $d_{\varphi} p$, then

$A \subseteq \mathcal{M}$, $|A| \leq |T|$.

By $|T|^{+}$-saturation $\exists b' \in \mathcal{M}$ st. $b' \equiv_{Ac} b$.

Then $\models d_{\varphi} p(b')$ & $\not\models d_{\varphi}' p(b')$ (because $\chi(b', c)$).

So $d_{\varphi} p$, $d_{\varphi}' p$ do not ~~def~~ define the same φ-type in $\mathcal{M}$.

(ii) Let B be any set. ~~[illegible]~~

We want to prove $p|_B$ is a complete consistent type.

~~consistent~~ Let A be, as above, the set of parameters.

consistent: if not, there are $\varphi_i(x_i, b_i) \in p|_B$ $i < n$ st.

$\bigwedge \varphi_i(x, b_i)$ is inconsistent.

By saturation, find $\bar{b}' \equiv_{A} \bar{b}$, $\bar{b}' \in \mathcal{M}$.

Then $\varphi_i(x, b_i') \in p$ $\forall i$ and $\bigwedge \varphi_i(x, b_i')$ is inconsistent

#

complete: Assume not. Then $\exists\, b \in B$ and $\psi(x, y)$ s.t.

$\psi(x, b) \notin p|_B$ and $\forall \psi$ contradicting φ, $\psi(x, b) \notin p|_B$.

find $b' \equiv_A b$ in $\mathcal{M}$ ....etc..

(iii) not enough time, so exercise!