Second proof of remark given last time:

Assume $x_0 \subseteq x_1 \subseteq x_2 \subseteq \cdots$ are increasing tuples of variables.

$p_0(x_0) \subseteq p_1(x_1) \subseteq \cdots$ are increasing <u>complete</u> types

Let $a_i \models p_i \ \forall i$

For each $i \leq j$, let $a_{j,i}$ be the subtuple of a_j corresponding to x_i.

Define $b_0 \subseteq b_1 \subseteq b_2 \subseteq \cdots$ s.t. $b_i \models p_i$

Let $b_0 = a_0$. Assuming we have b_i, the $tp(b_i) = tp(a_{i+1,i}) = p_i$

so there is an automorphism f sending $a_{i+1,i}$ to b_i.

Let $b_{i+1} = f(a_{i+1})$. Let $b = \bigcup b_i$. Then $b \models \bigcup p_i$.

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__Definition__ A partial type $p(x,b)$ __divides__ over c

(x, b, c are possibly infinite tuples of variables/elements of $\mathcal{U}$)

if there is an indiscernible sequence $(b_i : i<\omega)$ in $tp(b/c)$

s.t. $\bigwedge_{i<\omega} p(x,b_i)$ $[= \bigcup p(x,b_i)]$ is inconsistent.

__Remark:__ If $(b_i : i<\omega)$ is an indiscernible sequence in $tp(b/c)$ then it has an automorphic image which is also c-indiscernible.

__Proof__: ~~There is~~
By compactness, for every λ, there is a similar sequence$/\emptyset$ in $tp(b/c)$. [Let $p_n(x_0 \dots x_{n-1}) = tp(b_0 \dots b_{n-1})$: $\bigwedge_{i<\lambda} q(x_i) \wedge$
$q = tp(b/c)$
$\bigwedge_n \bigwedge_{i_0<\dots<i_n} p_n(x_{i_0} \dots x_{i_{n-1}})$
consistent.)

Now extract a c-indiscernible sequence: $(b_i'' : i<\omega)$
is b_i' the similar sequence in $tp(b/c)$ of length λ? yes

(1) $b_i' \models q$ for all $i \Rightarrow b_i'' \models q$,
(2) $\forall i_0 < \dots < i_{n-1}$ $\models p_n(b_{i_0}' \dots b_{i_{n-1}}') \Rightarrow \models p_n(b_0'' \dots b_{n-1}'')$

$\Rightarrow$ $tp(b_i : i<\omega) = tp(b_i'' : i<\omega)$.
$\Rightarrow$ $b_{<\omega}''$ is an automorphic image of $b_{<\omega}$. $\square$

__Definition__ $a \underset{c}{\downarrow} b$ (read: a independent of b over c)

if $tp(a/bc)$ does not divide$/c$.
and

Proposition. $a \mathop{\downarrow}_c b$ if and only if every c-indiscernible sequence in $tp(b/c)$ has an c-automorphic image in $tp(b/ac)$.

Proof: $\Rightarrow$: Assume $a \mathop{\downarrow}_c b$ i.e. $tp(a/bc)$ does not divide (dnd) over c.

write $p(x, bc) = tp(a/bc)$.

Let $(b_i : i<\omega)$ be an c-indiscernible sequence in $tp(b/c)$.

~~Then $(b_ic : i<\omega)$ is indiscernible in $tp(bc/c)$.~~

[By the remark, there is an automorphic image $(b_i' : i<\omega)$ which is c-indiscernible and in $tp(b/c)$.]

$\Rightarrow$ $(b_i'c : i<\omega)$ is indiscernible in $tp(bc/c)$.

Since $tp(a/bc)$ dnd over c, there is $a' \models \bigwedge p(x, b_i'c)$.

In particular, $a' \models tp(a/c)$.

~~Applying~~ Let f be an c-automorphism s.t. $f(a'c) = ac$.

Therefore $(b_i'') := f(b_i')$ is an c-automorphic image of (b_i)

and $\models \bigwedge p(a, b_i''c) \Rightarrow b_i'' \models tp(b/ac)$.

$\Leftarrow$: Let (b_ic_i) be any c-indiscernible sequence in $tp(bc/c)$ ($\Rightarrow c_i = c\ \forall i$)

~~In particular (b_i) is an indiscernible~~

We need to find $a' \models \bigwedge p(x, b_ic)$.

By assumption (b_i) has a c-automorphic image (b_i') in $tp(b/ac)$.

Let f be the c-automorphism & let $a' = f^{-1}(a)$.

Then $\bigwedge p(a, b_i'c) \Rightarrow \bigwedge p(a', b_i c)$. $\square$.

Corollary: (1). Downward right-hand transitivity:

$$\forall a, b, c, d: \quad a \underset{c}{\downarrow} b,d \Rightarrow a \underset{c}{\downarrow} b \wedge a \underset{bc}{\downarrow} d.$$

(2) Upward left-hand transitivity:

$$a \underset{c}{\downarrow} b \text{ and } d \underset{ac}{\downarrow} b \Rightarrow ad \underset{c}{\downarrow} b.$$

Proof (1) Assume $a \underset{c}{\downarrow} bd$ then if (b_i) is c-indiscernible in $tp(b/c)$ then by extension/extraction, we can find (d_i) st. $(b_i d_i)$ is c-indiscernible in $tp(bd/c)$. ie $tp(b_i d_i/c) = tp(bd/c)$.

(Extend to $(b_i : i<\lambda)$, for each i find d_i st. $b_i d_i \equiv_c bd$ (by aut sending b to b_i) and extract a c-indiscernible sequence $(b_i' d_i')$.

but $b_{<\omega} \equiv_c b'_{<\omega}$ (both are similar c-indiscernible ~~sequences~~ seq of same length.

so we may (WMA) assume $b_{<\omega} = b'_{<\omega}$.

Since $a \underset{c}{\downarrow} bd$, there is a c-automorphic image $(b_i' d_i')$ in $tp(bd/ac)$

in particular (b_i') is a c-automorphic image of (b_i) in $tp(b/ac) \Rightarrow a \underset{c}{\downarrow} b$.

Now: let (bcd_i) be bc-indiscernible in $tp(bcd/bc)$.

Then it is c-indiscernible in $tp(bcd/c)$.

Let $tp(a/bcd) = q(x\,bcd)$.

So $\bigwedge q(x, bcd_i)$ is consistent $\Rightarrow a \underset{bc}{\downarrow} d$.

(2) Assume $a \underset{c}{\downarrow} b$, $d \underset{ac}{\downarrow} b$.

Let (b_i) be a c-indiscernible sequence in $tp(b/c)$.

Since $a \underset{c}{\downarrow} b$ there is a c-automorphic image (b_i') of (b_i) in $tp(b/ac)$.

b_i' is c-indiscernible and by a previous remark has c-automorphic image which is still in $tp(b/ac)$ and in addition is ac-indiscernible.

So we may assume (b_i') is ac-indiscernible.

Since $d \underset{ac}{\downarrow} b$, then (b_i') has an ac-automorphic image in $tp(b/acd)$

Conclusion (b_i) has a c-automorphic image in $tp(b/acd)$

$\Rightarrow ad \underset{c}{\downarrow} b$ $\square$

Lemma A partial type $p(x,b)$ divides $/c$ iff there is a formula $\varphi(x,b) \in p(x,b)$ which does.

(convension: all partial types are closed under conjunction)

Proof $\Leftarrow$: clear.

$\Rightarrow$: Assume (b_i) is c-indiscernible and $\bigcup p(x,b_i)$ is inconsistent. By compactness, only finitely many

formulas are required for inconsistency, say
$\varphi_0(x, b_{i_0}) \in p(x, b_{i_0}), \ldots, \varphi_{k-1}(x, b_{i_{k-1}}) \in p(x, b_{i_{k-1}})$.
let $\psi = \bigwedge \varphi_i(x,y)$. Then $\psi(x,b) \in p(x,b)$.
and $\bigwedge \psi(x, b_i)$ is inconsistent $\Rightarrow \psi(x,b)$ divides $/c$. $\square$

<u>Corollary</u> <u>Finite Character</u> $a \mathop{\downarrow}_c b$ (a,b,c are possibly infinite)
$\iff \forall a' \subseteq a$ and $\forall b' \subseteq b$ finite, $a' \mathop{\downarrow}_c b'$

<u>Proof</u> $\Rightarrow$: Clear.
$\Leftarrow$: if $a \mathop{\not\downarrow}_c b$ then there is a formula $\varphi(x, bc) \in tp(a/bc)$
which divides over c.
now only finite subtuples ~~among~~ $x' \subseteq x$ and $b' \subseteq b$ actually appear in φ. Let $a' \subseteq a$ correspond to $x' \subseteq x$.
$\Rightarrow \varphi(x', b'c) \in tp(a'/b'c)$.
$\Rightarrow a' \mathop{\not\downarrow}_c b'$ $\square$.