18.997 Topics in Combinatorial Optimization 16 March 2004

Lecture 11

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Let $\mathcal{M}_1 = (S, \mathcal{I}_1), \mathcal{M}_2 = (S, \mathcal{I}_2)$ be two matroids on common ground set S with rank functions r_1 and r_2. Many combinatorial optimization problems can be reformulated as the problem of finding the maximum size common independent set $I \in \mathcal{I}_1 \cap \mathcal{I}_2$. This problem was studied by Edmonds and Lawler, who proved the following min-max matroid intersection characterization.

Theorem 1

$$\max_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} |I| = \min_{U \in S} \left(r_1(U) + r_2(S \setminus U) \right).$$

As with many min-max characterizations, proving one of the inequalities is straightforward. For any $U \subseteq S$ and $I \in \mathcal{I}_1 \cap \mathcal{I}_2$, we have

$$\begin{aligned} |I| &= |I \cap U| + |I \cap (S \setminus U)| \\ &\leq r_1(U) + r_2(S \setminus U), \end{aligned}$$

since $I \cap U$ is an independent set in $\mathcal{I}_1$ and $I \cap (S \setminus U)$ is an independent set in $\mathcal{I}_2$. Therefore, $\max_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} |I| \leq \min_{U \in S} (r_1(U) + r_2(S \setminus U))$.

The following important examples illustrate some of the applications of the matroid intersection theorem.

Examples

1. For a bipartite graph $G = (V, E)$ with color classes $V = V_1 \cup V_2$, consider $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$ where $\mathcal{I}_i = \{F : \forall v \in V_i, \deg_F(v) \leq 1\}$ for $i = 1, 2$. Note that $\mathcal{M}_1$ and $\mathcal{M}_2$ are (partition) matroids, while $\mathcal{I}_1 \cap \mathcal{I}_2$, the set of bipartite matchings of G, does not define a matroid on E. Also, note that the rank $r_i(F)$ of F in M_i is the number of vertices in V_i covered by edges in F. Then by Theorem 1, the size of a maximum matching in G is

$$\begin{aligned} \nu(G) &= \min_{U \in E} (r_1(U) + r_2(E \setminus U)) & (1) \\ &= \tau(G) & (2) \end{aligned}$$

   where $\tau(G)$ is the size of a minimum vertex cover of G. Thus, the matroid intersection theorem generalizes Kőnig's matching theorem.

2. As a corollary to Theorem 1, we have the following min-max relationship for the minimum common spanning set in two matroids.

$$\begin{aligned} \min_{F \text{ spanning in } M_1 \text{ and } M_2} |F| &= \min_{B_i \text{ basis in } \mathcal{M}_i} |B_1 \cup B_2| \\ &= \min_{B_i \text{ basis in } \mathcal{M}_i} |B_1| + |B_2| - |B_1 \cap B_2| \\ &= r_1(S) + r_2(S) - \min_{U \subseteq S} [r_1(U) + r_2(S \setminus U)]. \end{aligned}$$

   Applying this corollary to the matroids in example 1, it follows that the minimum edge cover in G is equal to the maximum of $|V| - r_1(F) - r_2(E \setminus F)$ over all $F \subseteq E$. Since this is exactly the maximum size of a stable set in G, the corollary is a generalization of the Kőnig-Rado theorem.

3. Consider a graph G with a k-coloring on the edges, i.e., edge set E is partitioned into color classes $E_1 \cup E_2 \cup \ldots \cup E_k$. The question of whether or not there exists a rainbow spanning tree (i.e. a spanning tree with edges of different colors) can be restated as a matroid intersection problem on $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$ with

$$\begin{aligned} \mathcal{I}_1 &= \{F \subseteq E : F \text{ is acyclic}\} \\ \mathcal{I}_2 &= \{F \subseteq E : |F \cap E_i| \leq 1 \ \forall i\} \end{aligned}$$

Since $\mathcal{I}_1 \cap \mathcal{I}_2$ is the set of rainbow forests, there is a rainbow spanning tree of G if and only if

$$\max_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} |I| = |V| - 1.$$

By Theorem 1, this is equivalent to the condition

$$\min_{U \subseteq E} (r_1(U) + r_2(E \setminus U)) = |V| - 1.$$

Since $r_1(U) = |V| - c(U)$ (where $c(U)$ denotes the number of connected components of (V, U)), it follows that there is a rainbow spanning tree of G if and only if the number of colors in $E \setminus U$ is at least $c(U) - 1$ for any subset $U \subseteq E$. In other words, a rainbow spanning tree exists if and only if removing the edges of any t colors leaves a graph with at most $t + 1$ components.

4. Given a digraph $G = (V, A)$, a branching D is a subset of arcs such that

   (a) D has no directed cycles

   (b) For every vertex $v, \deg_{\text{in}}(v) \leq 1$ in D.

   Branchings are the common independent sets of matroids $\mathcal{M}_1 = (E, \mathcal{I}_1), \mathcal{M}_2 = (E, \mathcal{I}_2)$, where

$$\begin{aligned} \mathcal{I}_1 &= \{F \subseteq E : F \text{ is acyclic in the underlying undirected graph } G\} \\ \mathcal{I}_2 &= \{F \subseteq E : \deg_{\text{in}}(v) \leq 1 \ \forall v \in V\} \end{aligned}$$

Note that $\mathcal{M}_1$ is a graphic matroid on G and $\mathcal{M}_2$ is a partition matroid. Therefore, the problem of finding a maximum branching of a digraph can be solved by the matroid intersection algorithm.

In order to prove Theorem 1, we need the following lemmas. Recall that a circuit is a minimal dependent set.

Lemma 2 *Let $M = (S, \mathcal{I})$ be a matroid. If $I \in \mathcal{I}, I + x \notin \mathcal{I}$, then $I + x$ contains a unique minimal circuit.*

Lemma 3 *(Basis exchange) Suppose B_1 and B_2 are two bases of a matroid $\mathcal{M}$. For any $x \in B_1 \setminus B_2$, there exists $y \in B_2 \setminus B_1$ such that*

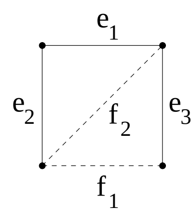
$$B_1 - x + y \in \mathcal{I} \text{ and } B_2 - y + x \in \mathcal{I}$$

Given an independent set I in a matroid $\mathcal{M} = (S, \mathcal{I})$, we define a digraph with vertex set S and arc set $A_M(I) = \{(x, y) : x \in I, y \in S \setminus I, I - x + y \in \mathcal{I}\}$. We often drop the M subscript when referring to A. This digraph plays a crucial role in several matroid optimization algorithms including matroid intersection.

Lemma 4 *Let $I, J \in \mathcal{I}$ with $|I| = |J|$. Then $A(I)$ contains a matching on $I \Delta J = (I \setminus J) \cup (J \setminus I)$.*

Proof: We can assume I, J are bases in $\mathcal{I}$ (otherwise, consider the truncated matroid whose independent sets are those in $\mathcal{I}$ of size less than or equal to $|I|$). We proceed by induction on $|I \setminus J|$. For any $x \in I \setminus J$, there exists $y \in J \setminus I$ such that $J' = J - y + x \in \mathcal{I}$. Then $I \setminus J' = (I \setminus J) - x$ and $J' \setminus I = (J \setminus I) - y$. If $|I \setminus J| = 1$, then we are done; otherwise by induction on $|I \setminus J|$, $A(I)$ contains a matching on $I \Delta J'$, which we extend to a matching of $I \Delta J$ by adding edge (x, y). □

Unfortunately, the converse of this theorem is not true, as shown by the following counterexample. Let $\mathcal{M}$ be the graphic matroid on the following graph G.



For $I = \{e_1, e_2, e_3\}, J = \{f_1, f_2, e_3\}$, $A(I)$ contains a matching $(e_1, f_1), (e_2, f_2)$ of $I \Delta J$ and $I \in \mathcal{I}$, but $J \notin \mathcal{I}$.

However, by a slight strengthening of the condition, we can prove the following.

Lemma 5 *Given matroid $\mathcal{M} = (S, \mathcal{I}), I \in \mathcal{I}$, and $J \subseteq S$ with $|I| = |J|$, if $A(I)$ contains a* unique *matching on $I \Delta J$, then $J \in \mathcal{I}$.*

Note that in the example above, $A(I)$ also contains the matching $(e_1, f_2), (e_2, f_1)$ on $I \Delta J$, so the stronger condition fails.

Proof: Let N denote the unique perfect matching on $I \Delta J$ and consider the digraph in which we reverse the orientation of the arcs in N. By the uniqueness of the perfect matching, there are no directed cycles in the resulting graph, so there is a topological ordering of the vertices. This ordering induces a labeling on vertices in $N = \{(y_1, z_1), (y_2, z_2), \dots (y_t, z_t)\}$ such that there are no arcs (y_i, z_j) for $i < j$.

If $J \notin \mathcal{I}$, then it contains a circuit C. Let i be the smallest index such that $z_i \in C$. Since there are no arcs from y_i to z_j with $j > i$, $I - y_i + z_j \notin \mathcal{I}$, implying $z_j \in \text{span}(I - y_i)$. Since this is true for all $j > i, C - z_i \subseteq \text{span}(I - y_i)$. But since C is a circuit, $z_i \in \text{span}(C - z_i) \subseteq \text{span}(I - y_i)$. Then $I - y_i + z_i \notin \mathcal{I}$ and by definition of $A(I)$, $(y_i, z_i) \notin A(I)$ (since $I - y_i + z_i \notin \mathcal{I}$), a contradiction to the existence of perfect matching N. Therefore $J \in \mathcal{I}$. □

Now, we state the matroid intersection algorithm, whose proof we will give in the next lecture. Since $\mathcal{I}$ may be exponential in size, we assume our matroid is described by an oracle which, given $I \subseteq S$, can determine in polynomial time if $I \in \mathcal{I}$. Then the running time of the algorithm is polynomial in the number of calls to the oracle.

First, for $I \subseteq S$, define the digraph $D(I) = (S, A)$ as follows: for $y \in I$, $x \notin I$, we have an arc $(y, x) \in A$ if $I - y + x \in \mathcal{I}_1$ and $(x, y) \in A$ if $I - y + x \in \mathcal{I}_2$. This is the union of the arcset $A_{M_1}(I)$ corresponding to $\mathcal{I}_1$ and the reverse of the arcset $A_{M_2}(I)$ corresponding to $\mathcal{I}_2$. Consider the sets

$$X_1 = \{x \in S \setminus I : I + x \in \mathcal{I}_1\}, X_2 = \{x \in S \setminus I : I + x \in \mathcal{I}_2\}.$$

Matroid Intersection Algorithm

Input Matroids $\mathcal{M}_1 = (S, \mathcal{I}_1)$, $\mathcal{M}_2 = (S, \mathcal{I}_2)$

Output $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ of maximum size

$I \leftarrow \emptyset$

while $D(I)$ has a path from X_1 to X_2

$I \leftarrow I \Delta V(P)$, where P is a *shortest* path from X_1 to X_2.

We will prove the correctness of this algorithm in the next lecture.