**18.997 Topics in Combinatorial Optimization** March 30, 2004

# Lecture 13

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Last lecture we covered matroid intersection, and defined matroid union. In this lecture we review the definitions of matroid intersection, and then show that the matroid intersection polytope is TDI. This is Chapter 41 in Schrijver's book. Next we review matroid union, and show that unlike matroid intersection, the union of two matroids is again a matroid. This material is largely contained in Chapter 42 in Schrijver's book. We leave testing independence in the union matroid for the next lecture.

## 1 Matroid Intersection

Matroid intersection is defined for two matroids on the same ground set, $M_1 = (S, \mathcal{I}_1)$, $M_2 = (S, \mathcal{I}_2)$. In the last lecture, we saw that the size of the largest independent set in the intersection is given by:

$$\max_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} |I| = \min_{U \subseteq S} \{r_1(U) + r_2(S \setminus U)\},$$

where $r_1$ ($r_2$) is the rank function of the first (second) matroid. Also in last lecture, we defined the matroid intersection polytope:

$$\mathcal{P} \triangleq \left\{ \begin{array}{lllr} x(U) & \leq & r_1(U) & \forall U \subseteq S \\ x(U) & \leq & r_2(U) & \forall U \subseteq S \\ x & \geq & 0 & \end{array} \right\}.$$

**Theorem 1** *The polytope $\mathcal{P}$ defined above is totally dual integrable (TDI).*

In fact more is true. Schrijver shows that $\mathcal{P}$ is Box-TDI, which means that $\mathcal{P}$ is TDI, and so is $\mathcal{P} \cap \{x : l_i \leq x_i \leq u_i\}$, for any integral lower and upper bounds $l_i, u_i \in \mathbb{Z}$.

**Proof:** We need to show that for all choices of integral weight function $w \in \mathbb{Z}^n$, the dual of the LP

$$\begin{array}{ll} \max: & w^T x \\ \text{s.t.}: & x \in \mathcal{P}, \end{array}$$

is integral. The dual is given by

$$\begin{array}{ll} \min: & \displaystyle\sum_{U \subseteq S} y_1(U) r_1(U) + \sum_{U \subseteq S} y_2(U) r_2(U) \\ \text{s.t.}: & \displaystyle\sum_{U : i \in U} (y_1(U) + y_2(U)) \leq w_i, \quad \forall i \\ & y_1(U), y_2(U) \geq 0. \end{array}$$

Recall that a matrix $A$ is called totally unimodular (TUM) if and only if any square submatrix $B$ is such that $\det(B) \in \{0, \pm 1\}$. If an LP is defined by a TUM matrix $A$, then it must be integral. The matrix that defines the dual above is not, however, TUM. We show that we can restrict the dual, setting a subset of the variables to zero, still obtaining an equivalent formulation. We show that in this equivalent, restricted formulation, the defining matrix is in fact totally unimodular.

Let the optimum value of the dual be attained at the point $(y_1^*, y_2^*)$. The first component of the solution, $y_1^*$, can be regarded as the optimal solution to the problem

$$\begin{aligned}
\text{min :} \quad & \sum_{U \subseteq S} y_1(U) r_1(U) \\
\text{s.t. :} \quad & \sum_{U : i \in U} y_1(U) \leq w_i - \sum_{U : i \in U} y_2^*(U) \\
& y_1(U) \geq 0.
\end{aligned}$$

This is the dual of a maximum independent set problem in $M_1$, with weight vector $\hat{w}$, where $\hat{w}_i = w_i - \sum_{U:i \in U} y_2^*(U)$. In Lecture 11 (and also in Schrijver, Chapter 40.2) we saw that the greedy algorithm optimally solves maximum independent set problems in matroids. The greedy algorithm orders the elements of the ground set in non-increasing order, according to the weight function $\hat{w}$. Then, letting $U_i := \{s_1, \ldots, s_i\}$, the greedy algorithm can be used to exhibit a dual solution where $y_1(U_i) = \hat{w}(s_i) - \hat{w}(s_{i+1})$, and $y_1(U) = 0$ for $U \neq U_i$ for some $i$. Let $\mathcal{F}_1$ denote the sets $U$ of the above form. Note that $\mathcal{F}_1$ is a (nested) chain of sets. Therefore, using the greedy algorithm, we can assume that given $y_2^*$, the corresponding problem in $y_1$ has an optimal solution $y_1^*$ that satisfies $y_1^*(U) = 0$ for $U \notin \mathcal{F}_1$.

Similarly, for any fixed $y_1^*$, the resulting problem in $y_2$ can be solved as the dual to a maximum independent set problem, and therefore there is a nested chain of subsets $\mathcal{F}_2$, such that there exists an optimal solution $y_2^*$ with $y_2^*(U) = 0$ for $U \notin \mathcal{F}_2$.

Therefore we have shown that the dual problem above is equivalent to the restriction

$$\begin{aligned}
\text{min :} \quad & \sum_{U \subseteq S} y_1(U) r_1(U) + \sum_{U \subseteq S} y_2(U) r_2(U) \\
\text{s.t. :} \quad & \sum_{U : i \in U} (y_1(U) + y_2(U)) \leq w_i, \quad \forall\, i \\
& y_1(U) = 0, \quad \forall\, U \notin \mathcal{F}_1 \\
& y_2(U) = 0, \quad \forall\, U \notin \mathcal{F}_2 \\
& y_1(U), y_2(U) \geq 0.
\end{aligned}$$

The important point is that $\mathcal{F}_1$ and $\mathcal{F}_2$ are sets of nested subsets of the ground set $S$. Let $A$ denote the nonzero columns of the matrix defining the restricted dual problem above. The next theorem says that the restricted matrix $A$ is in fact totally unimodular. This implies that the dual problem is integral, and hence concludes the proof of the theorem. $\square$

We have left to prove that the matrix $A$ above is in fact totally unimodular. First, we give a definition:

**Definition 1** *A collection of sets $\mathcal{F}$ is called laminar if $A, B \in \mathcal{F}$ implies that $A \subseteq B$, $B \subseteq A$, or $A \cap B = \emptyset$.*

**Theorem 2** *If $\mathcal{F}$ is the union of two laminar families of subsets of a set $X$, then the $X \times \mathcal{F}$ incidence matrix of $\mathcal{F}$, is totally unimodular.*

First, note that since $\mathcal{F}_1$ and $\mathcal{F}_2$ each are a nested chain of subsets, they are both laminar, and hence the matrix $A$ indeed satisfies the hypotheses of the theorem. Also note that the theorem fails if $\mathcal{F}$ is the union of *three* laminar families. For (a somewhat trivial) example, we have

$$\det\left(\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\right) = -2.$$

The matrix is the incidence matrix of the union of three laminar families (each laminar family contains only one set), yet the determinant of the matrix is $-2$, and thus it cannot be totally unimodular.

**Proof:** Let $A$ be our matrix, which is the incidence matrix of a set $\mathcal{F}$, which is the union of two laminar families: $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$. Let $B$ be any square submatrix. We need to show that $\det(B) \in \{0, \pm 1\}$. The columns of $A$ each correspond to an element of the family $\mathcal{F}$.

Consider now the columns of matrix $B$. Some of them come from $\mathcal{F}_1$, and others from $\mathcal{F}_2$. Consider any two columns $C_1, C_2$ of $B$ (or the sets corresponding to them) such that both are elements of one of the two families $\mathcal{F}_i$ and $C_1 \subseteq C_2$. By replacing $C_2$ by the componentwise difference, $C_2 - C_1$, we can at most change the sign of the determinant. Repeating this procedure for all pairs of columns of $B$ coming from $\mathcal{F}_1$, and then for all columns coming from $\mathcal{F}_2$, we obtain a matrix $\hat{B}$, whose determinant has the same magnitude as the determinant of $B$. In addition all columns corresponding to $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) correspond to disjoint sets.

The matrix $\hat{B}$ has at most 2 one's in each row. If there exists a row with no one's, then $\det(B) = 0$ and we are done. If there exists a row with a single one, then the proof follows by induction, since we can expand by minors about that entry, thus reducing the size of the matrix we are considering. Finally, if all rows have two ones, then by the construction of $\hat{B}$, the sum of the columns from $\mathcal{F}_1$ must equal the sum of the columns from $\mathcal{F}_2$, and hence $\det(\hat{B}) = \det(B) = 0$. This concludes the proof of the theorem. $\square$

## 2 Matroid Union

We saw in the last lecture that the intersection of two matroids (on the same ground set) need not be a matroid (but nevertheless had nice properties). Consider, for example, the ground set $\{a, b, c\}$, and the two matroids given by the independent sets $\mathcal{I}_1 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}$ and $\mathcal{I}_2 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$. The intersection is not a matroid.

In this section, we show that the union of matroids is again a matroid. Then, take matroids, $M_1 = (S_1, \mathcal{I}_1), \ldots, M_k = (S_k, \mathcal{I}_k)$. The union matroid is defined as $M = (S, \mathcal{I})$, where $S \stackrel{\triangle}{=} S_1 \cup \cdots \cup S_k$, and

$$\mathcal{I} \stackrel{\triangle}{=} \mathcal{I}_1 \vee \cdots \vee \mathcal{I}_k = \{I_1 \cup \cdots \cup I_k \,:\, I_i \in \mathcal{I}_i, i = 1, \ldots, k\}.$$

**Theorem 3** *The union matroid $M = (S, \mathcal{I})$ as given above, is indeed a matroid.*

When the ground sets are disjoint, it is straightforward to see that $M$ is in fact a matroid. For the case where the ground sets $S_i$ are not all disjoint, we use the following lemma.

**Lemma 4** *Given any matroid $M' = (S', \mathcal{I}')$, and any function (not necessarily injective) $f : S' \to S$, then $M = (S, f(\mathcal{I}'))$ is a matroid, where*

$$f(\mathcal{I}') = \{f(I') \,:\, I' \in \mathcal{I}'\}.$$

**Proof:** Since $f$ is a function, it is clear that if $I \in f(\mathcal{I}')$, then any subset of $I$ is also in $f(\mathcal{I}')$. Now suppose $I, J \in f(\mathcal{I}')$, with $|I| < |J|$. We need to show that for some $j \in J \setminus I$, $I + j \in f(\mathcal{I}')$. By assumption (and definition) $I$ and $J$ must be images of two independent sets $I', J'$ of $M'$. Since $f$ is not injective, there may be many ways to choose such sets. We take $I', J'$ so that $I = f(I')$ and $|I| = |I'|$, $J = f(J')$ and $|J| = |J'|$, and finally, such that $|I' \cap J'|$ is maximal.

Since $|I'| < |J'|$ and $M'$ is, by assumption, a matroid, then there exists an element $t \in J' \setminus I'$ such that $I + t \in \mathcal{I}'$. If $f(t) \in f(I') \cap f(J')$, then there exists some $u \in I'$ such that $f(t) = f(u)$. Since $|J'| = |J|$, $f$ maps $J'$ injectively onto $J$, and thus $u \in I' \setminus J'$. But then the set $I'' = I' - u + t \in \mathcal{I}$ (because $I' + t \in \mathcal{I}$), $f(I'') = I$, $|I''| = |I|$, and $|I'' \cap J'| > |I' \cap J'|$ contradicting maximality. Therefore $f(t) \in f(J') \setminus f(I')$, and $f(I' + t) = f(I') + f(t) \in f(\mathcal{I})$, as required. $\square$

The proof of the matroid union theorem follows quickly from the lemma.

**Proof:** Let $\{S'_i\}_{i=1}^k$ be disjoint copies of the original ground sets $S_i$, and $\mathcal{I}'_i$ the corresponding independent sets. Let $S' = S'_1 \cup \cdots \cup S'_k$, and let $\mathcal{I}' = \{I' = I'_1 \cup \cdots \cup I'_k : I'_i \in \mathcal{I}'_i\}$. Then $M' = (S', \mathcal{I}')$ is a matroid. For $S$ the union (not disjoint) of the ground sets of the matroids, we have a map $f : S \to S'$. The union matroid $M = (S, \mathcal{I})$ is the image of the matroid $M'$ above, under the map $f$. The above lemma now applies directly, and we conclude that $M = (S, \mathcal{I}) = (S, f(\mathcal{I}'))$ is indeed a matroid. □

Next we determine the rank function of the union matroid. Again we consider the more general setup of the lemma above. Under the same definitions, we have:

**Lemma 5** *If the rank function of $M' = (S', \mathcal{I}')$ is $r'$, then the rank function of the matroid $M = (S, f(\mathcal{I}'))$ is given by*

$$r(U) = \min_{T \subseteq U}(|U \setminus T| + r'(f^{-1}(T))).$$

**Proof:** The independent sets of $M$ are the images of independent sets in $M'$. Therefore the size of the largest independent set $I \subset U$ in $M$, is the size of the largest independent set $I' \in f^{-1}(U)$ in $M'$, that maps injectively to $I = f(I')$. Therefore we are asking for the size of the largest common independent set in in $M'$ and in the partition matroid we obtain from the inverse mapping $f^{-1}$. Recall that for two matroids $M_1, M_2$ on the same ground set $S$, we have the formula

$$\max_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} |I| = \min_{U \subseteq S}\{r_1(U) + r_2(S \setminus U)\}.$$

Thus we have

$$r(U) = \min_{T \subseteq U}(|U \setminus T| + r'(f^{-1}(T))).$$

□

Applying this result to the union matroid, we find that the rank function is given by

$$r_{\text{union}}(U) = \min_{T \subseteq U}(|U \setminus T| + r_1(T \cap S_1) + \cdots + r_k(T \cap S_k)),$$

for $U \subseteq S_1 \cup \cdots \cup S_k$.

## 3 Next Lecture

We still have not discussed how we might actually test independence in a union matroid. To see that this is not a trivial problem, consider, for example, the matroid whose independent sets are the forests of a graph. We can consider the union matroid on $k$ copies of the graph. Now a set will be independent if it is the union of $k$ forests. Given such a union, how can we determine if a given edge $e$ may be added in order to obtain a larger independent set? Even given an explicit decomposition of the union into $k$ forests, this is a nontrivial problem, since the given decomposition need not be unique. This is one of the issues addressed in the next lecture.