18.997 Topics in Combinatorial Optimization February 26th, 2004

Lecture 6

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Last time, we saw that the matching polytope was defined by:

$$x(\delta(v)) \le 1 \quad \forall v \in V$$
$$x(E(S)) \le \left\lfloor \frac{|S|}{2} \right\rfloor, \text{ for } |S| \text{ odd}$$
$$x \ge 0.$$

One may wonder whether we need all *blossom* inequalities $x(E(S)) \le \left\lfloor \frac{|S|}{2} \right\rfloor$. In other words, which of these inequalities define facets of the polytope and are essential in the description.

Theorem 1 *$x(E(S)) \le \lfloor \frac{|S|}{2} \rfloor$ is necessary in the description of the matching polytope iff $G[S]$ is factor-critical and 2-connected.*

The proof is in the book, Ch.25-5. The fact that they are necessary uses the Theorem mentioned in lecture 3 that the number of affinely independent near-perfect matchings in a 2-connected factor-critical graph is equal to $|E|$.

1 Partially Ordered Sets (posets) — Ch.14

Definition 1 *S is poset with relation $\le$ when,*

- $s \le s$
- $s \le t$ *and* $t \le s \Rightarrow s = t$
- $s \le t$ *and* $t \le v \Rightarrow s \le v$ *(transitive)*

for $s, t, v \in S$.

$s < t$ means $s \le t$ and $s \ne t$. The poset S induces a digraph (S,A) such that A is the set of edges (s,t) if $s < t$.

Definition 2 *An antichain A is a subset of S such that $\forall s \ne t \in A, t \not< s$ and $s \not< t$. A chain C is a subset of S such that $\forall s \ne t \in C, s \le t$ or $t \le s$.*

We define the maximum s of a chain C as the element of C such that $t \le s$ for all $t \in C$. The maximum element exists and is unique in any chain of a poset.

We can easily see that $|A \cap C| \le 1$ for any antichain A and any chain C.

Theorem 2 $\max_C |C|$ *= minimum number of antichains A_i's which partition S.*

Theorem 3 *(Dilworth's theorem)* $\max_A |A|$ *= minimum number of chains C_j's which partition S.*

In both theorems 2 and 3, " $\le$" part is clear by $|A \cap C| \le 1$. For both theorems, it would be enough to prove the existence of the appropriate number of chains or antichains *covering* S rather than partitioning it.

Proof: (proof of theorem 2 "$\ge$" part) Define *height*(s) as the number of elements in the longest chain whose maximum is s. Then the maximum of all heights equals to $\max_C |C|$. Let $A_i =$

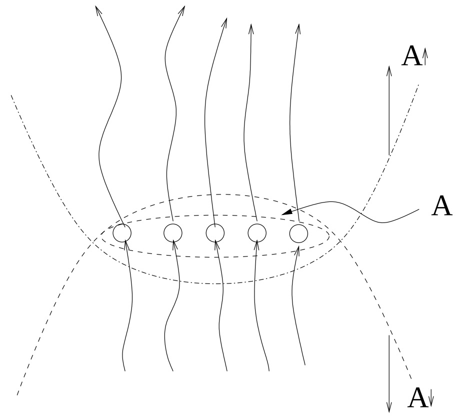

Figure 1: A maximum antichain A and the subgraph $A^{\uparrow}$ and $A^{\downarrow}$.

$\{s|height(s) = i\}$, then A_i is an antichain. $A_1, A_2, \cdots, A_M$ is a set of antichains which partitions S where M=$\max_C |C|$.

□

Proof: (proof of theorem 3 "$\geq$" part) Take a maximum size antichain A and $|A| = \alpha$. Let $A^{\uparrow} = \{s|\exists t \in A,\ t \leq s\}$ and $A^{\downarrow} = \{s|\exists t \in A,\ s \leq t\}$.

We consider two cases.

- Case 1: there is no antichain A of maximum size such that $A \neq A^{\uparrow}$ and $A \neq A^{\downarrow}$. In this case, the only antichains of maximum size are either the set of maximal elements or the set of minimal elements. Choose a minimal element s and a maximal element t satisfying $s \leq t$ (this is always possible otherwise we could find a larger antichain). Think of $S \setminus \{s,t\}$. Then the maximum size of an antichain decreases because, otherwise, there would exist an antichain A such that $A \neq A^{\uparrow}$ and $A \neq A^{\downarrow}$. So the maximum size of an antichain decreases by 1 and it is $\alpha - 1$. By induction, $S \setminus \{s,t\}$ can be partitioned by $\alpha - 1$ chains. Adding the chain $\{s,t\}$, S can be partitioned by α chains.

- Case 2: there is an antichain A of maximum size such that $A \neq A^{\uparrow}$ and $A \neq A^{\downarrow}$.

  A is a maximum antichain in both $A^{\uparrow}$, $A^{\downarrow}$. By induction on the size of the poset, there exists α chains in each set $A^{\uparrow}$, $A^{\downarrow}$ which partition the poset. Now we merge every two chains, one from each set, which intersect. That will give α chains which partition the original set.

□

2 Bipartite Matching

Dilworth's theorem is actually equivalent to König's theorem for bipartite graphs. Let us start by stating König's theorem and showing it follows from our discussion on non-bipartite matching.

Let $\nu(G)$ be the size of a maximum matching in G and $\tau(G)$ be the size of a minimum vertex cover in G.

Theorem 4 *(König's theorem) $\nu(G) = \tau(G)$ in a bipartite graph G.*

We'll prove this using the Edmonds-Gallai structure since we have already proved it, but this is an overkill. König's theorem can be proved quite simply.

Proof: The following sets are defined in the Edmonds-Gallai structure.

- $D(G)$ = vertices missed by some matching,
- $A(G) = N(D(G))$ = the neighborhood of $D(G)$ and
- $C(G) = V(G) \setminus D(G) \setminus A(G)$.

We already know that each component of $G[D(G)]$ is factor-critical and that each component of $G[C(G)]$ has a perfect macthing. So we can find a vertex cover which has the same size as the maximum matching as follows.

1. Take all of $A(G)$.
2. Each component of $C(G)$ is bipartite and contains a perfect matching, so we can take one side of the bipartition within $C(G)$ in the vertex cover. Here we have selected half of the vertices of $C(G)$.
3. From each component of $D(G)$, delete one of the vertices which is adjacent to a vertex of $A(G)$. Then each component of $D(G)$ turns to be of even size and we can take half of the elements to put within the vertex cover.

Combining all the vertices we got from 1, 2, 3, we obtain a vertex cover of G of size:

$$|A(G)| + \frac{|C(G)|}{2} + \frac{|D(G)| - o(G \setminus A(G))}{2} = \frac{1}{2}[|V| + |A(G)| - o(G \setminus A(G))],$$

where $o(G \setminus A(G))$ is the number of odd components of $G \setminus A(G)$. Therefore we found a vertex cover which has the same size as a maximum matching.

□

Now we will show a different proof of Dilworth's theorem using König's theorem.

Proof: (Dilworth's theorem) For a poset S, define a bipartite graph H with $2n$ vertices by creating two vertices a_i, b_i for each $i \in S$, and the edge set $E(H)$ is such that $(a_s, b_t) \in E(H) \Leftrightarrow s < t$.

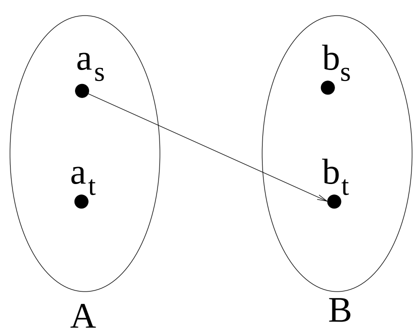

Figure 2: The bipartite graph H.

We claim that a matching M in the graph H corresponds to a partitioning of S by $n - |M|$ chains. Indeed, just consider the arcs the poset that correspond to the edges in the matching. Every vertex i of S can have at most one incoming arc (if b_i is matched) and one outgoing arc (if a_i is matched). This means that it corresponds to a partitioning of S by chains. The number of chains is equal to the number of vertices with no incoming edge and this is equal to the number of unmatched vertices in B, i.e. $n - |M|$.

Now consider a minimum vertex cover F of H. Let $\hat{A} = \{s \in S | a_s \notin F \text{ } and \text{ } b_s \notin F\}$. The fact that F is a vertex cover implies that $\hat{A}$ is an antichain. Furthermore, we have that $|\hat{A}| \geq n - |F| = n - |M|$, which is the number of chains we have obtained in the first part that partitions S. Therefore we proved the $\geq$ part of Theorem 3. □

3 Weighted posets

We can generalize Theorems 2 and 3 to weighted posets. Let $w : S \to Z_+$, be a weight function defined on the elements of S.

Theorem 5 $\max_C w(C)(= \sum_{s \in C} w_s)$ *where the maximum is taken over the chains C is equal to the minimum number of antichains which cover each s w(s) times,* $\forall s \in S$.

Theorem 6 $\max_A w(A)(= \sum_{s \in A} w_s)$ *where the maximum is taken over the antichains A is equal to the minimum number of chains which cover each s w(s) times,* $\forall s \in S$.

Both theorems have to be stated in terms of covering S instead of partitioning S.

Proof: (Proof of Theorem 5) Replace a vertex s by a chain of w(s) copies of s, $\forall s \in S$. If $s \leq t$ in the original poset, we have that $s^{(i)} \leq t^{(j)}$ when $s^{(i)}$ (resp. $t^{(j)}$) is any copy of s (resp. t). See Figure 3. Let the resulting poset denoted by S'. Then apply Dilworth's theorem to S'. A maximum cardinality chain in S' corresponds to a chain in S of maximum weight. Also a minimum covering of S' by antichains corresponds to a covering of S by antichains which cover s $w(s)$ times.

□

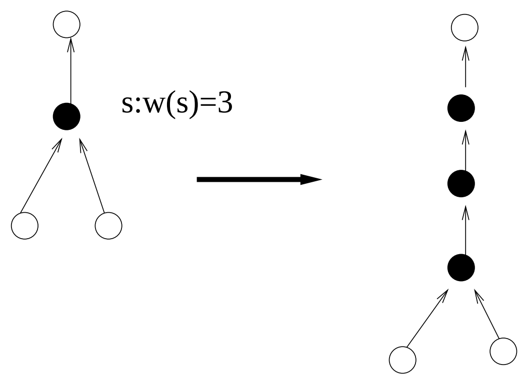

Figure 3: Making $w(s)$ chain copies of s.

Proof: (Proof of Theorem 6) Replace a vertex s by an antichain of w(s) copies of s, $\forall s \in S$. By applying Dilworth's theorem to the resulting poset will give the proof by similar reasoning to the previous proof. See Figure 4.

□

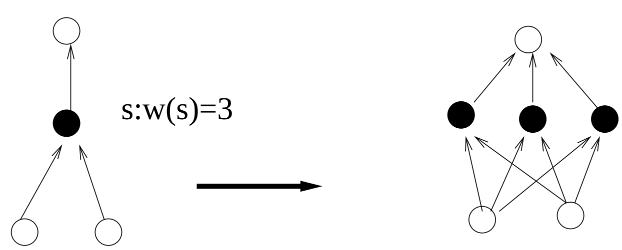

Figure 4: Making w(s) incomparable copies of s.

4 Chain Polytope

Theorem 5 is equivalent to the fact that the following system is TDI:

$Max \sum w_s x_s$
s.t. $x(A) \leq 1$, $\forall$ antichain A
$x_s \geq 0$, $s \in S$.

The dual is :
$Min \sum_A y_A$
s.t. $\sum_{(A:s \in A)} y_A \geq w_s$
$y_A \geq 0$.

From Theorem 5, we derive that both the primal and the dual have an integer optimal solution whenever w_s is integral for all s. This means that the system is TDI.

The following antichain polytope is TDI for the same reason.
$Max \sum w_s z_s$
s.t. $z(C) \leq 1$, $\forall$ antichain C
$z_s \geq 0$, $s \in S$.