18.S096: Homework Problem Set 4

Topics in Mathematics of Data Science (Fall 2015)

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Connectivity of the Erdős-Rényi random graph

The Erdős-Rényi random graph $G(n,p)$ is a graph with n nodes, where each edge (i,j) appears (independently) with probability p. In this problem set, you will show a remarkable phase transition: if $\lambda < 1$, then $G(n, \frac{\lambda \ln n}{n})$ has, with high probability, isolated nodes while, if $\lambda > 1$, the graph is connected (with high probability).

Problem 4.1 *Let I_i be a random variable indicating whether node i is isolated: $I_i = 1$ if node i is isolated, and $I_i = 0$ otherwise. Let $X = \sum_{i=1}^{n} I_i$ be the number of isolated nodes.*

The goal is to show that $\Pr\{X = 0\}$ is small when $\lambda < 1$ (meaning that there are isolated notes, with high probability). In the proof you can use the approximation

$$(1 - \lambda/n)^n \approx e^{-\lambda} \quad \textit{(for large } n\textit{)}$$

1. *Show that $\mathbb{E}[X] \approx n^{-\lambda+1}$.* ***Note:*** *The fact that $\mathbb{E}[X] \to \infty$ is not sufficient to show $\Pr\{X = 0\} \to 0$ (**why? Can you give a counter-example?**). We need to ensure that X concentrates around its mean.*

2. *Use (a simple) concentration inequality derived in class to finish the proof. (The techinque you have just derived is known as the second moment method)*

Problem 4.2 *Prove that, if $\lambda \geq 1$, $G(n, \frac{\lambda \ln n}{n})$ is connected with high probability:*

1. *Derive the probability for a set of k nodes ($k \leq n/2$) being disconnected from the rest of the graph.*

2. *Prove the probability of graph G having a disconnected component goes to zero as n grows (hint: use union bound).*