18.S096: Homework Problem Set 5

Topics in Mathematics of Data Science (Fall 2015)

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Problem 5.1 (Little Grothendieck problem) *Let $C \succeq 0$ (C is positive semidefinite). In this homework you'll show an approximation ratio of $\frac{2}{\pi}$ to the problem*

$$\max_{x_i=\pm 1} \sum_{i,j=1}^{n} C_{ij} x_i x_j.$$

Similarly to `Max-Cut`, we consider

$$\max_{\substack{v_i \in \mathbb{R}^n \\ \|v_i\|^2=1}} \sum_{i,j=1}^{n} C_{ij} v_i^T v_j.$$

The goal is to show that, for $r \sim \mathcal{N}(0, I_{n\times n})$, taking $x_i^\natural = \operatorname{sign}(v_i^T r)$ a randomized rounding,

$$\mathbb{E}\left[\sum_{i,j=1}^{n} C_{ij} x_i^\natural x_j^\natural\right] \geq \frac{2}{\pi} \sum_{i,j=1}^{n} C_{ij} v_i^T v_j$$

Hints:

1. *The main difficulty is that $\mathbb{E}\left[\operatorname{sign}(v_i^T r)\operatorname{sign}(v_j^T r)\right]$ is not linear in $v_i^T v_j$ and C_{ij} might be negative for some (i,j)'s.*
2. *Show that that $\mathbb{E}\left[\operatorname{sign}(v_i^T r) v_j^T r\right]$ is linear in $v_i^T v_j$. What is it equal to?*
3. *Construct S with entries $S_{ij} = \left(v_i^T r - \sqrt{\frac{2}{\pi}}\operatorname{sign}(v_i^T r)\right)\left(v_j^T r - \sqrt{\frac{2}{\pi}}\operatorname{sign}(v_j^T r)\right)$*
4. *Show that $\operatorname{Tr}(CS) \geq 0$.*

Problem 5.2 *Consider the problem of angular synchronization. Suppose we have n clocks. Between any two clocks, we observe the relative hourly difference. The goal is to tell the time on each clock, up* *to a global time shift. Let* $z = [e^{i\theta_1}, \dots, e^{i\theta_n}]^T \in \mathbb{C}^n$ *represents the ground truth time vector we try to recover (*$\theta_k \in [0, 2\pi], k = 1, \dots, n$ *are effectively the time on the clocks). Then the true pairwise time difference between clock* k *and* l *can be represented by*

$$z_k z_l^* = e^{i(\theta_k - \theta_l)}$$

.

Consider A_{kl} *the observed time difference between clock* k *and* l*.* A_{kl} *equals* $z_k z_l^*$ *with probability* p *and* $A_{kl} = e^{i\phi}$ *with probability* $1 - p$*.* ϕ *is a random uniform variable on the interval* $[0, 2\pi]$*.*

1. *Since we know that when* p *is small* $A_{kl} \approx z_k z_l^*$*, a way to solve the synchronization problem is by solving*

$$\min_u \sum_{k=1}^n \sum_{l=1}^n |u_k u_l^* - A_{kl}|^2, \quad s.t.\ |u_k| = 1,\ k = 1, \dots, n$$

   Relax this to a problem of computing the eigenvector associated with the maximum eigenvalue of a matrix.

2. *The goal is to understand when the eigenvector method returns meaningful solution. Without loss of generality we can assume* $z_i = 1$ *for all* i*.*

   - *In that case what is* $\mathbb{E}A$*?*
   - *Are the entries of* $A - \mathbb{E}A$ *independent (and identically distributed)?*
   - *If the entries of* A *were Gaussian (but with the same mean and variance they have), argue that*

$$\frac{p}{\sqrt{1 - p^2}} > \frac{1}{\sqrt{n}}$$

     is needed so that the eigenvector method returns a meaningful solution.

Remark: *It turns out that the fact that the entries are not Gaussian, in this case, does not drastically affect the behavior of the top eigenvector and so this prediction is very accurate. Also, the condition is not only needed but sufficient.*

Problem 5.3 ($\mathbb{Z}_2$ **Synchronization)** *Consider a similar synchronization-type problem. Let* $z \in \{-1, 1\}^n$ *denote the faces of* n *coins (or a clock that only gives noon and 6 o'clock). From the noisy observations* A_{ij} *of the relative faces between the* n *coins (or the relative time) and we want to solve*

$$\min_{x \in \{-1,1\}^n} \sum_{i=1}^n \sum_{j=1}^n |x_i x_j - A_{ij}|^2$$

where

$$A_{ij} = z_i z_j + \sigma W,$$

where $W_{ij} \sim \mathcal{N}(0, 1)$ *with independent entries (expect for the fact that* $W^T = W$*).*

The purpose is to understand for which values of σ *is a Semidefinite programming relaxation exact.*

1. *Show that this problem is equivalent to*

$$\max_{x\in\{-1,1\}^n} \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}x_ix_j$$

2. *Derive an SDP relaxation for this problem.*

3. *Show that*

$$\min \operatorname{Tr}(D) \qquad \text{such that } D \text{ is diagonal, and } D - A \succeq 0$$

   satisfies weak duality: for any feasible D, the objective of this program is at least the one of the original SDP.

4. *Without Loss of generality take $z_i = 1$ for all i.*

5. *Take $D = D_A$ where $(D_A)_{ii} = \sum_{j=1}^{n} A_{ij}$. When does this choice of D is able to prove that 11^T is an optimal solution of the original SDP? For which values of σ do we expect this to happen with high probability? Can it also be shown that 11^T is the unique solution? (Such a D is known as a dual certificate or a dual witness).*

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