0.2.2 Matrix AM-GM inequality

We move now to an interesting generalization of arithmetic-geometric means inequality, which has applications on understanding the difference in performance of with- versus without-replacement sampling in certain randomized algorithms (see [RR12]).

Open Problem 0.2 *For any collection of $d \times d$ positive semidefinite matrices $A_1, \cdots, A_n$, the following is true:*

(a)

$$\left\| \frac{1}{n!} \sum_{\sigma \in \mathrm{Sym}(n)} \prod_{j=1}^{n} A_{\sigma(j)} \right\| \leq \left\| \frac{1}{n^n} \sum_{k_1, \ldots, k_n = 1} \prod_{j=1}^{n} A_{k_j} \right\|,$$

and

(b)

$$\frac{1}{n!} \sum_{\sigma \in \mathrm{Sym}(n)} \left\| \prod_{j=1}^{n} A_{\sigma(j)} \right\| \leq \frac{1}{n^n} \sum_{k_1, \ldots, k_n = 1}^{n} \left\| \prod_{j=1}^{n} A_{k_j} \right\|,$$

where $\mathrm{Sym}(n)$ *denotes the group of permutations of* n *elements, and* $\|\cdot\|$ *the spectral norm.*

Morally, these conjectures state that products of matrices with repetitions are larger than without. For more details on the motivations of these conjecture (and their formulations) see [RR12] for conjecture **(a)** and [Duc12] for conjecture **(b)**.

Recently these conjectures have been solved for the particular case of $n = 3$, in [Zha14] for **(a)** and in [IKW14] for **(b)**.

References

[Duc12] J. C. Duchi. Commentary on "towards a noncommutative arithmetic-geometric mean inequality" by b. recht and c. re. 2012.

[IKW14] A. Israel, F. Krahmer, and R. Ward. An arithmetic-geometric mean inequality for products of three matrices. *Available online at arXiv:1411.0333 [math.SP]*, 2014.

[RR12] B. Recht and C. Re. Beneath the valley of the noncommutative arithmetic-geometric mean inequality: conjectures, case-studies, and consequences. *Conference on Learning Theory (COLT)*, 2012.

[Zha14] T. Zhang. A note on the non-commutative arithmetic-geometric mean inequality. *Available online at arXiv:1411.5058 [math.SP]*, 2014.