1.2.1 A related open problem

Open Problem 1.2 (Monotonicity of singular values [BKS13a]) *Consider the setting above but with $p = n$, then $X \in \mathbb{R}^{n\times n}$ is a matrix with iid $\mathcal{N}(0,1)$ entries. Let*

$$\sigma_i\left(\frac{1}{\sqrt{n}}X\right),$$

denote the i-th singular value[4] *of $\frac{1}{\sqrt{n}}X$, and define*

$$\alpha_{\mathbb{R}}(n) := \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}\sigma_i\left(\frac{1}{\sqrt{n}}X\right)\right],$$

as the expected value of the average singular value of $\frac{1}{\sqrt{n}}X$.

The conjecture is that, for every $n \geq 1$,

$$\alpha_{\mathbb{R}}(n+1) \geq \alpha_{\mathbb{R}}(n).$$

Moreover, for the analogous quantity $\alpha_{\mathbb{C}}(n)$ defined over the complex numbers, meaning simply that each entry of X is an iid complex valued standard gaussian $\mathbb{C}\mathcal{N}(0,1)$ the reverse inequality is conjectured for all $n \geq 1$:

$$\alpha_{\mathbb{C}}(n+1) \leq \alpha_{\mathbb{C}}(n).$$

Notice that the singular values of $\frac{1}{\sqrt{n}}X$ are simply the square roots of the eigenvalues of S_n,

$$\sigma_i\left(\frac{1}{\sqrt{n}}X\right) = \sqrt{\lambda_i\left(S_n\right)}.$$

[4] The i-th diagonal element of Σ in the SVD $\frac{1}{\sqrt{n}}X = U\Sigma V$.

This means that we can compute $\alpha_{\mathbb{R}}$ in the limit (since we know the limiting distribution of $\lambda_i\left(S_n\right)$) and get (since $p=n$ we have $\gamma=1$, $\gamma_-=0$, and $\gamma_+=2$)

$$\lim_{n\to\infty}\alpha_{\mathbb{R}}(n)=\int_0^2\lambda^{\frac{1}{2}}dF_1(\lambda)=\frac{1}{2\pi}\int_0^2\lambda^{\frac{1}{2}}\frac{\sqrt{(2-\lambda)\,\lambda}}{\lambda}=\frac{8}{3\pi}\approx 0.8488.$$

Also, $\alpha_{\mathbb{R}}(1)$ simply corresponds to the expected value of the absolute value of a standard gaussian g

$$\alpha_{\mathbb{R}}(1)=\mathbb{E}|g|=\sqrt{\frac{2}{\pi}}\approx 0.7990,$$

which is compatible with the conjecture.

On the complex valued side, the Marchenko-Pastur distribution also holds for the complex valued case and so $\lim_{n\to\infty}\alpha_{\mathbb{C}}(n)=\lim_{n\to\infty}\alpha_{\mathbb{R}}(n)$ and $\alpha_{\mathbb{C}}(1)$ can also be easily calculated and seen to be larger than the limit.

Reference

[BKS13a] A. S. Bandeira, C. Kennedy, and A. Singer. Approximating the little grothendieck problem over the orthogonal group. *Available online at arXiv:1308.5207 [cs.DS]*, 2013.