

10.2 Angular Synchronization

The angular synchronization problem [Sin11, BSS13] consist in estimating n unknown angles $\theta_1, \dots, \theta_n$ from m noisy measurements of their offsets $\theta_i - \theta_j \pmod{2\pi}$. This problem easily falls under the scope of synchronization-type problem by taking a graph with a node for each θ_i , an edge associated with each measurement, and taking the group to be $\mathcal{G} \cong SO(2)$, the group of in-plane rotations. Some of its applications include time-synchronization of distributed networks [GK06], signal reconstruction from phaseless measurements [ABFM12], surface reconstruction problems in computer vision [ARC06] and optics [RW01].

Let us consider a particular instance of this problem (with a particular noise model).

Let $z_1, \dots, z_n \in \mathbb{C}$ satisfying $|z_a| = 1$ be the signal (angles) we want to estimate ($z_a = \exp(i\theta_a)$). Suppose for every pair (i, j) we make a noisy measurement of the angle offset

$$Y_{ij} = z_i \bar{z}_j + \sigma W_{ij},$$

where $W_{ij} \sim \mathcal{N}(0, 1)$. The maximum likelihood estimator for z is given by solving (see [Sin11, BBS14])

$$\max_{|x_i|=1} x^* Y x. \tag{103}$$

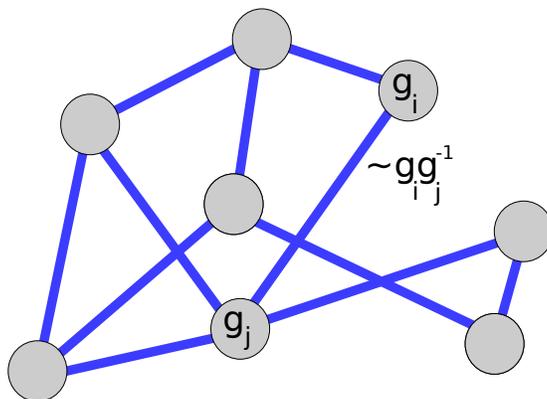


Figure 22: Given a graph $G = (V, E)$ and a group \mathcal{G} , the goal in synchronization-type problems is to estimate node labels $g : V \rightarrow \mathcal{G}$ from noisy edge measurements of offsets $g_i g_j^{-1}$.

There are several approaches to try to solve (103). Using techniques very similar to the study of the spike model in PCA on the first lecture one can (see [Sin11]), for example, understand the performance of the spectral relaxation of (103) into

$$\max_{\|x\|^2=n} x^* Y x. \tag{104}$$

Notice that, since the solution to (104) will not necessarily be a vector with unit-modulus entries, a rounding step will, in general, be needed. Also, to compute the leading eigenvector of A one would likely use the power method. An interesting adaptation to this approach is to round after each iteration of the power method, rather than waiting for the end of the process, more precisely:

Algorithm 10.1 *Given Y . Take a original (maybe random) vector $x^{(0)}$. For each iteration k (until convergence or a certain number of iterations) take $x^{(k+1)}$ to be the vector with entries:*

$$\left(x^{(k+1)}\right)_i = \frac{(Y x^{(k)})_i}{|(Y x^{(k)})_i|}.$$

Although this method appears to perform very well in numeric experiments, its analysis is still an open problem.

Open Problem 10.1 *In the model where $Y = zz^* + \sigma W$ as described above, for which values of σ will the Projected Power Method (Algorithm 10.1) converge to the optimal solution of (103) (or at least to a solution that correlates well with z), with high probability?*³⁷

References

- [ARC06] A. Agrawal, R. Raskar, and R. Chellappa. What is the range of surface reconstructions from a gradient field? In A. Leonardis, H. Bischof, and A. Pinz, editors, *Computer Vision – ECCV 2006*, volume 3951 of *Lecture Notes in Computer Science*, pages 578–591. Springer Berlin Heidelberg, 2006.
- [ABFM12] B. Alexeev, A. S. Bandeira, M. Fickus, and D. G. Mixon. Phase retrieval with polarization. *available online*, 2012.
- [BSS13] A. S. Bandeira, A. Singer, and D. A. Spielman. A Cheeger inequality for the graph connection Laplacian. *SIAM J. Matrix Anal. Appl.*, 34(4):1611–1630, 2013.
- [BBS14] A. S. Bandeira, N. Boumal, and A. Singer. Tightness of the maximum likelihood semidefinite relaxation for angular synchronization. *Available online at arXiv:1411.3272 [math.OA]*, 2014.
- [GK06] A. Giridhar and P.R. Kumar. Distributed clock synchronization over wireless networks: Algorithms and analysis. In *Decision and Control, 2006 45th IEEE Conference on*, pages 4915–4920. IEEE, 2006.
- [RW01] J. Rubinstein and G. Wolansky. Reconstruction of optical surfaces from ray data. *Optical Review*, 8(4):281–283, 2001.
- [Sin11] A. Singer. Angular synchronization by eigenvectors and semidefinite programming. *Appl. Comput. Harmon. Anal.*, 30(1):20 – 36, 2011.

³⁷We thank Nicolas Boumal for suggesting this problem.

MIT OpenCourseWare
<http://ocw.mit.edu>

18.S096 Topics in Mathematics of Data Science
Fall 2015

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.