10.3.2 The semidefinite relaxation

We will now present a semidefinite relaxation for (108) (see [BCSZ14]).

Let us identify R_l with the $L \times L$ permutation matrix that cyclicly permutes the entries fo a vector by l_i coordinates:

$$R_l \begin{bmatrix} u_1 \\ \vdots \\ u_L \end{bmatrix} = \begin{bmatrix} u_{1-l} \\ \vdots \\ u_{L-l} \end{bmatrix}.$$

This corresponds to an L-dimensional representation of the cyclic group. Then, (108) can be rewritten:

$$\begin{aligned} \sum_{i,j\in[n]} \langle R_{-l_i} y_i, R_{-l_j} y_j \rangle &= \sum_{i,j\in[n]} \left(R_{-l_i} y_i\right)^T R_{-l_j} y_j \\ &= \sum_{i,j\in[n]} \operatorname{Tr}\left[\left(R_{-l_i} y_i\right)^T R_{-l_j} y_j\right] \\ &= \sum_{i,j\in[n]} \operatorname{Tr}\left[y_i^T R_{-l_i}^T R_{-l_j} y_j\right] \\ &= \sum_{i,j\in[n]} \operatorname{Tr}\left[\left(y_i y_j^T\right)^T R_{l_i} R_{l_j}^T\right]. \end{aligned}$$

We take

$$X = \begin{bmatrix} R_{l_1} \\ R_{l_2} \\ \vdots \\ R_{l_n} \end{bmatrix} \begin{bmatrix} R_{l_1}^T & R_{l_2}^T & \cdots & R_{l_n}^T \end{bmatrix} \in \mathbb{R}^{nL \times nL}, \tag{109}$$

and can rewrite (108) as

$$\begin{array}{ll} \max & \operatorname{Tr}(CX) \\ \text{s. t.} & X_{ii} = I_{L\times L} \\ & X_{ij} \text{ is a circulant permutation matrix} \\ & X \succeq 0 \\ & \operatorname{rank}(X) \leq L, \end{array} \tag{110}$$

where C is the rank 1 matrix given by

$$C = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \begin{bmatrix} y_1^T & y_2^T & \cdots & y_n^T \end{bmatrix} \in \mathbb{R}^{nL \times nL}, \tag{111}$$

with blocks $C_{ij} = y_i y_j^T$.

The constraints $X_{ii} = I_{L\times L}$ and $\mathrm{rank}(X) \le L$ imply that $\mathrm{rank}(X) = L$ and $X_{ij} \in O(L)$. Since the only doubly stochastic matrices in $O(L)$ are permutations, (110) can be rewritten as

$$\begin{array}{ll} \max & \mathrm{Tr}(CX) \\ \text{s. t.} & X_{ii} = I_{L\times L} \\ & X_{ij}\mathbf{1} = \mathbf{1} \\ & X_{ij} \text{ is circulant} \\ & X \ge 0 \\ & X \succeq 0 \\ & \mathrm{rank}(X) \le L. \end{array} \tag{112}$$

Removing the nonconvex rank constraint yields a semidefinite program, corresponding to (**??**),

$$\begin{array}{ll} \max & \mathrm{Tr}(CX) \\ \text{s. t.} & X_{ii} = I_{L\times L} \\ & X_{ij}\mathbf{1} = \mathbf{1} \\ & X_{ij} \text{ is circulant} \\ & X \ge 0 \\ & X \succeq 0. \end{array} \tag{113}$$

Numerical simulations (see [BCSZ14, BKS14]) suggest that, below a certain noise level, the semidefinite program (113) is tight with high probability. However, an explanation of this phenomenon remains an open problem [BKS14].

Open Problem 10.3 *For which values of noise do we expect that, with high probability, the semidefinite program* (113) *is tight? In particular, is it true that for any σ by taking arbitrarily large n the SDP is tight with high probability?*

References

[BCSZ14] A. S. Bandeira, M. Charikar, A. Singer, and A. Zhu. Multireference alignment using semidefinite programming. *5th Innovations in Theoretical Computer Science (ITCS 2014)*, 2014.

[BKS14] A. S. Bandeira, Y. Khoo, and A. Singer. Open problem: Tightness of maximum likelihood semidefinite relaxations. In *Proceedings of the 27th Conference on Learning Theory*, volume 35 of *JMLR W&CP*, pages 1265–1267, 2014.