Definition 5.13 (The Restricted Isometry Property) *An $M \times N$ matrix A (with either real or complex valued entries) is said to satisfy the (s, δ)-Restricted Isometry Property (RIP),*

$$(1-\delta)\|x\|^2 \leq \|Ax\|^2 \leq (1+\delta)\|x\|^2,$$

for all s-sparse x.

Using Proposition 5.12 and Theorem 5.7 one can readily show that matrices with Gaussian entries satisfy the restricted isometry property with $M \approx s \log\left(\frac{N}{s}\right)$.

Theorem 5.14 *Let A be an $M \times N$ matrix with i.i.d. standard gaussian entries, there exists a constant C such that, if*

$$M \geq Cs \log\left(\frac{N}{s}\right),$$

then $\frac{1}{a_M}A$ satisfies the $\left(s, \frac{1}{3}\right)$-RIP, with high probability.

Theorem 5.14 suggests that RIP matrices are abundant for $s \approx \frac{M}{\log(N)}$, however it appears to be very difficult to deterministically construct matrices that are RIP for $s \gg \sqrt{M}$, known as the square bottleneck [Tao07, BFMW13, BFMM14, BMM14, B$^+$11, Mix14a]. The only known unconditional construction that is able to break this bottleneck is due to Bourgain et al. [B$^+$11] that achieves $s \approx M^{\frac{1}{2}+\varepsilon}$ for a small, but positive, ε. There is a conditional construction, based on the Paley Equiangular Tight Frame, that will be briefly described in the next Lecture [BFMW13, BMM14].

Open Problem 5.1 *Construct deterministic matrices $A \in \mathbb{C}^{M\times N}$ (or $A \in \mathbb{C}^{M\times N}$) satisfying $(s, \frac{1}{3})$-RIP for $s \gtrsim \frac{M^{0.6}}{\mathrm{polylog}(N}$.*

References

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