6.5.2 Equiangular Tight Frames

Another natural question is whether one can get better coherence (or more vectors) by relaxing the condition that the set of vectors have to be formed by taking orthonormal basis. A tight frame (see, for example, [CK12] for more on Frame Theory) is a set of N vectors in $\mathbb{C}^M$ (with $N \geq M$) that spans $\mathbb{C}^M$ "equally". More precisely:

Definition 6.6 (Tight Frame) *$v_1, \ldots, v_N \in \mathbb{C}^M$ is a tight frame if there exists a constant α such that*

$$\sum_{k=1}^{N} |\langle v_k, x\rangle|^2 = \alpha \|x\|^2, \quad \forall_{x \in \mathbb{C}^M},$$

or equivalently

$$\sum_{k=1}^{N} v_k v_k^T = \alpha I.$$

The smallest coherence of a set of N unit-norm vectors in M dimensions is bounded below by the Welch bound (see, for example, [BFMW13]) which reads:

$$\mu \geq \sqrt{\frac{N-M}{M(N-1)}}.$$

One can check that, due to this limitation, deterministic constructions based on coherence cannot yield matrices that are RIP for $s \gg \sqrt{M}$, known as the square-root bottleneck [BFMW13, Tao07].

There are constructions that achieve the Welch bound, known as Equiangular Tight Frames (ETFs), these are tight frames for which all inner products between pairs of vectors have the same modulus $\mu = \sqrt{\frac{N-M}{M(N-1)}}$, meaning that they are "equiangular". It is known that for there to exist an ETF in $\mathbb{C}^M$ one needs $N \leq M^2$ and ETF's for which $N = M^2$ are important in Quantum Mechanics, and known as SIC-POVM's. However, they are not known to exist in every dimension (see here for some recent computer experiments [SG10]). This is known as Zauner's conjecture.

Open Problem 6.3 *Prove or disprove the SIC-POVM / Zauner's conjecture: For any d, there exists an Equiangular tight frame with d^2 vectors in $\mathbb{C}^d$ dimensions. (or, there exist d^2 equiangular lines in $\mathbb{C}^d$).*

We note that this conjecture was recently shown [Chi15] for $d = 17$ and refer the reader to this interesting remark [Mix14c] on the description length of the constructions known for different dimensions.

References

[BFMW13] A. S. Bandeira, M. Fickus, D. G. Mixon, and P. Wong. The road to deterministic matrices with the restricted isometry property. *Journal of Fourier Analysis and Applications*, 19(6):1123–1149, 2013.

[CK12] P. G. Casazza and G. Kutyniok. *Finite Frames: Theory and Applications.* 2012.

[Chi15] T.-Y. Chien. *Equiangular lines, projective symmetries and nice error frames.* PhD thesis, 2015.

[Mix14c] D. G. Mixon. Short, Fat matrices BLOG: Gordon's escape through a mesh theorem. 2014.

[SG10] A. J. Scott and M. Grassl. Sic-povms: A new computer study. *J. Math. Phys.*, 2010.

[Tao07] T. Tao. What's new blog: Open question: deterministic UUP matrices. 2007.

MIT OpenCourseWare
http://ocw.mit.edu

18.S096 Topics in Mathematics of Data Science
Fall 2015

For information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms.