7.3.1 Shannon Capacity

The goal Shannon Capacity is to measure the amount of information that can be sent through a noisy channel where some pairs of messages may be confused with eachother. Given a graph G (called the confusion graph) whose vertices correspond to messages and edges correspond to messages that may be confused with each other. A good example is the following: say one has a alphabet of five symbols $1, 2, 3, 4, 5$ and that each digit can be confused with the immediately before and after (and 1 and 5 can be confused with eachother). The confusion graph in this case is C_5, the cyclic graph

on 5 nodes. It is easy to see that one can at most send two messages of one digit each without confusion, this corresponds to the independence number of C_5, $\alpha(C_5) = 2$. The interesting question arises when asking how many different words of two digits can be sent, it is clear that one can send at least $\alpha(C_5)^2 = 4$ but the remarkable fact is that one can send 5 (for example: "11","23","35","54", or "42"). The confusion graph for the set of two digit words $C_5^{\oplus 2}$ can be described by a product of the original graph C_5 where for a graph G on n nodes $G^{\oplus 2}$ is a graph on n nodes where the vertices are indexed by pairs ij of vertices of G and

$$(ij, kl) \in E\left(G^{\oplus 2}\right)$$

if both a) $i = k$ or $i, k \in E$ and b) $j = l$ or $j, l \in E$ hold.

The above observation can then be written as $\alpha\left(C_5^{\oplus 2}\right) = 5$. This motivates the definition of Shannon Capacity [Sha56]

$$\theta_S\left(G\right) \sup_k \left(G^{\oplus k}\right)^{\frac{1}{k}}.$$

Lovasz, in a remarkable paper [Lov79], showed that $\theta_S\left(C_5\right) = \sqrt{5}$, but determining this quantity is an open problem for many graphs of interested [AL06], including C_7.

Open Problem 7.3 *What is the Shannon Capacity of the 7 cycle?*

References

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