8.4 The Grothendieck Constant

There is a somewhat similar remarkable problem, known as the Grothendieck problem [AN04, AMMN05]. Given a matrix $A \in \mathbb{R}^{n \times m}$ the goal is to maximize

$$\begin{array}{ll} \max & x^T A y \\ s.t. & x_i = \pm, \forall_i \\ s.t. & y_j = \pm, \forall_j \end{array} \tag{76}$$

Note that this is similar to problem (66). In fact, if $A \succeq 0$ it is not difficult to see that the optimal solution of (76) satisfies $y = x$ and so if $A = L_G$, since $L_G \succeq 0$, (76) reduces to (66). In fact, when $A \succeq 0$ this problem is known as the little Grothendieck problem [AN04, CW04, BKS13a].

Even when A is not positive semidefinite, one can take $z^T = [x^T\, y^T]$ and the objective can be written as

$$z^T \left[\begin{array}{cc} 0 & A \\ A^T & 0 \end{array} \right] z.$$

Similarly to the approximation ratio in Max-Cut, the Grothendieck constant [Pis11] K_G is the maximum ratio (over all matrices A) between the SDP relaxation

$$\begin{array}{ll} \max & \sum_{ij} A_{ij} u_i^T v_j \\ s.t. & u_i \in \mathbb{R}^{n+m}, \|u_i\| = 1, \\ & v_j \in \mathbb{R}^{n+m}, \|v_j\| = 1 \end{array} \tag{77}$$

and 76, and its exact value is still unknown, the best known bounds are available here [] and are $1.676 < K_G < \frac{\pi}{2\log(1+\sqrt{2})}$. See also page 21 here [F+14]. There is also a complex valued analogue [Haa87].

Open Problem 8.3 *What is the real Grothendieck constant K_G?*

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