18.S34 PROBLEMS #12

Fall 2007

127. [1] Find all solutions in integers to (a) $x + y = xy$, (b) $x + y + 1 = xy$, (c) $x^2 + y^2 = xy + x + y$.

128. [1.5] Choose 23 people at random. What is the probability some two of them have the same birthday? (You may ignore the existence of February 29.)

129. [1] Let p and q be consecutive odd primes (i.e., no prime numbers are between them). Show that $p + q$ is a product of at least three primes. For instance, $23 + 29$ is the product of the three primes 2, 2, and 13.

130. [3.5] Evaluate in closed form:

$$\int \frac{x\,dx}{\sqrt{x^4 + 4x^3 - 6x^2 + 4x + 1}}.$$

131. [1.5] Suppose that for each $n \geq 1$, $f_n(x)$ is a continuous function on the closed interval $[0, 1]$. Suppose also that for any $x \in [0, 1]$,

$$\lim_{n \to \infty} f_n(x) = 0.$$

Is it then true that

$$\lim_{n \to \infty} \int_0^1 f_n(x)dx = 0?$$

132. Let $x \geq 1$ be a real number. Let $f(x)$ be the maximum number of 1×1 squares that can fit inside an $x \times x$ square without overlap. (It is *not* assumed that the sides of the 1×1 squares are parallel to the sides of the $x \times x$ square.) For instance, if x is an integer then $f(x) = x^2$.

    (a) [3] Show that for some values of x, $f(x) > \lfloor x \rfloor^2$, where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$.

    (b) [5] Find a formula for (or at least a method of computing) $f(x)$ for any x.

133. [3] Let S be any finite set of points in the plane such that not all of them lie on a single straight line. Show that some (infinite) line intersects exactly two points of S.

134. [2.5] (The non-messing-up-theorem) Let M be an $m \times n$ matrix of integers. For example,

$$M = \begin{bmatrix} 7 & 3 & 1 & 4 & 2 \\ 5 & 6 & 3 & 1 & 5 \\ 2 & 2 & 1 & 8 & 4 \end{bmatrix}.$$

Rearrange the rows of M in increasing order.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 7 \\ 1 & 3 & 5 & 5 & 6 \\ 1 & 2 & 2 & 4 & 8 \end{bmatrix}.$$

Now rearrange the columns in increasing order.

$$\begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 4 & 7 \\ 1 & 3 & 5 & 5 & 8 \end{bmatrix}.$$

Show that the rows remain in increasing order.

135. (a) [2.5] Let $a(n)$ be the number of ways to write the positive integer n as a sum of *distinct* positive integers, where the order of the summands is not taken into account. Similarly let $b(n)$ be the number of ways to write n as a sum of *odd* positive integers, without regard to order. For instance, $a(7) = 5$, since $7 = 6+1 = 5+2 = 4+3 = 4+2+1$; while $b(n) = 5$, since $1+1+1+1+1+1+1 = 3+1+1+1+1 = 3+3+1 = 5+1+1 = 7$. Show that $a(n) = b(n)$ for all n.

(b) [3] Let A and B be subsets of the positive integers. Let $a_A(n)$ be the number of ways to write n as a sum (without regard to order) of distinct elements of the set A. Let $b_B(n)$ be the number of ways to write n as a sum (without regard to order) of elements of B. Call (A, B) an *Euler pair* if $a_A(n) = b_B(n)$ for all n. For instance, (a) above states that if A consists of all positive integers and B consists of the odd positive integers, then (A, B) is an Euler pair. Show that (A, B) is an Euler pair if and only if $2A \subseteq A$ (i.e., if $k \in A$ then $2k \in A$) and $B = A - 2A$.

(c) [1.5] Note that according to (b), if $A = \{1, 2, 4, 8, \ldots, 2^m, \ldots\}$ and $B = \{1\}$, then (A, B) is an Euler pair. What familiar fact is this equivalent to?

(d) [3.5] Let $c(n)$ denote the number of ways to write n as a sum (without regard to order) of positive integers, such that any two of the summands differ by at least two. Let $d(n)$ denote the number of ways to write n as a sum (without regard to order) of positive integers of the form $5k-1$ and $5k+1$, where k is an integer. For instance, $c(10)=5$, since $10=8+2=7+3=6+4=6+3+1$; while $d(10)=5$, since $1+1+1+1+1+1+1+1+1+1=4+1+1+1+1+1+1=6+1+1+1+1=4+4+1+1=6+4$. Show that $c(n)=d(n)$ for all n.

136. [2.5] Let

$$f(n)=\sum a_1a_2\cdots a_k,$$

where the sum is over all 2^{n-1} ways of writing n as an ordered sum $a_1+\cdots+a_k$ of positive integers a_i. For instance,

$$f(4)=4+3\cdot 1+1\cdot 3+2\cdot 2+2\cdot 1\cdot 1+1\cdot 2\cdot 1+1\cdot 1\cdot 2+1\cdot 1\cdot 1\cdot 1=21.$$

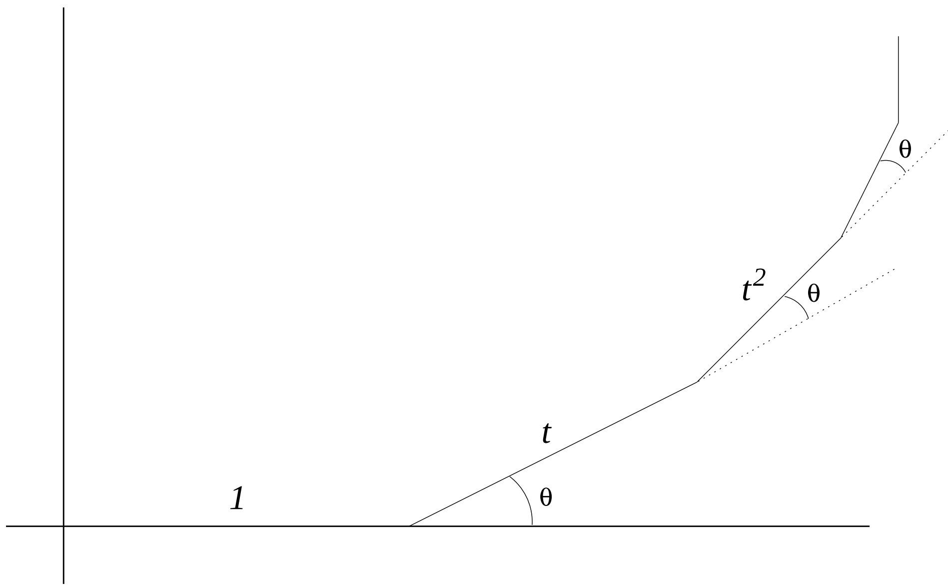
Find a simple expression for $f(n)$ in terms of Fibonacci numbers.

137. [2] Let $0^\circ \leq \theta \leq 180^\circ$ and $0<t<1$. A person stands at the origin in the (x,y)-plane and steps a distance of 1 in the positive x-direction. He then turns an angle θ counterclockwise and steps a distance t. He again turns θ counterclockwise and steps t^2. Continuing in this way, at the nth step he turns θ and steps a distance of t^{n-1}. As n increases, he will approach a limiting point $f(\theta,t)$ in the (x,y)-plane. For instance,

$$f(0^\circ,t)=(1+t+t^2+\cdots,0)=(1/(1-t),0)$$

$$f(180^\circ)=(1-t+t^2-t^3+\cdots,0)=(1/(1+t),0).$$

Find a simple formula for $f(\theta,t)$.



138. [2.5] Let x be a positive real number. Find the maximum value of the product $\prod_{i \in S} i$, where S is any subset of the positive real numbers whose sum is x. (HINT: First show that if the number k of elements of S is *fixed*, then maximum is achieved by taking all the elements of S to be equal to x/k. Then find the best value of k. For most numbers, k will be unique. But for each $k \geq 1$, there is an exceptional number x_k such that there are two sets S and S' which achieve the maximum, one with k elements and one with $k+1$ elements.)

139. [4] A polynomial $f(x) \in \mathbb{C}[x]$ is *indecomposable* if whenever $f(x) = r(s(x))$ for polynomials $r(x), s(x)$, then either $\deg r(x) = 1$ or $\deg s(x) = 1$. Suppose that $f(x)$ and $g(x)$ are nonconstant indecomposable polynomials in $\mathbb{C}[x]$ such that $f(x) - g(y)$ factors in $\mathbb{C}[x, y]$. (A trivial example is $x^2 - y^2 = (x-y)(x+y)$.) Show that either $g(x) = f(ax+b)$ for some $a, b \in \mathbb{C}$, or else

$$\deg f(x) = \deg g(x) = 7,\ 11,\ 13,\ 15,\ 21,\ \text{or } 31.$$

Moreover, this result is best possible in the sense that for $n = 7, 11, 13$, 15, 21, or 31, there exist indecomposable polynomials $f(x)$, $g(x)$ of degree n such that $f(x) \neq g(ax+b)$ and $f(x) - g(y)$ factors.