18.S34 PROBLEMS #3

Fall 2007

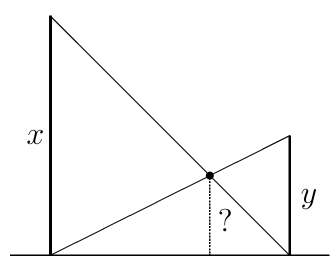
28. [1] Let $x, y > 0$. The *harmonic mean* of x and y is defined to be $2xy/(x+y)$. The *geometric mean* is $\sqrt{xy}$. The *arithmetic mean* (or *average*) is $(x+y)/2$. Show that

$$\frac{2xy}{x+y} \leq \sqrt{xy} \leq \frac{x+y}{2},$$

with equality if and only if $x = y$.

29. [1] A car travels one mile at a speed of x mi/hr and another mile at y mi/hr. What is the average speed? What kind of mean of x and y is this?

30. [1] Consider two telephone poles of heights x and y. Connect the top of each pole to the bottom of the other with a rope. What is the height of the point where the ropes cross? What kind of mean is this related to?



31. [1] Given two line segments of lengths x and y, describe a simple geometric construction for constructing a segment of length $\sqrt{xy}$.

32. [1] Suppose x and y are real numbers such that $x^2 + y^2 = x + y$. What is the largest possible value of x?

33. [2.5] (a) Let $x, y > 0$ and $p \neq 0$. The *p-th power mean* of x and y is defined to be

$$M_p(x,y) = \left(\frac{x^p + y^p}{2}\right)^{1/p}.$$

Note that $M_{-1}(x, y)$ is the harmonic mean and $M_1(x, y)$ is the arithmetic mean. If $p < q$, then show that

$$M_p(x, y) \le M_q(x, y),$$

with equality if and only if $x = y$.

(b) Compute $\lim_{p\to\infty} M_p(x, y)$, $\lim_{p\to 0} M_p(x, y)$, $\lim_{p\to-\infty} M_p(x, y)$.

34. [3.5] Let $x, y > 0$. Define two sequences $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ as follows:

$$\begin{aligned} x_1 &= x & y_1 &= y \\ x_{n+1} &= \frac{x_n + y_n}{2} & y_{n+1} &= \sqrt{x_n y_n}, \quad n > 1. \end{aligned}$$

It's not hard to see that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$. This limit is denoted $AG(x, y)$ and is called the *arithmetic-geometric mean* of x and y. Show that

$$AG(x, y) = \frac{\pi}{\displaystyle\int_0^\pi \frac{d\theta}{\sqrt{x^2 \sin^2\theta + y^2\cos^2\theta}}}.$$

35. Let $x > 0$. Define

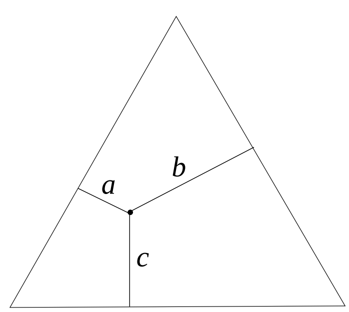
$$f(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}.$$

More precisely, let $x_1 = x$ and $x_{n+1} = x^{x_n}$ if $n > 1$, and define $f(x) = \lim_{n\to\infty} x_n$.

(a) [1] Compute $f(\sqrt{2})$.

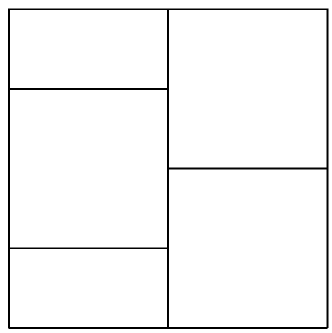
(b) [3.5] For what values of x does $f(x)$ exist?

36. [2] Let T be an equilateral triangle. Find all points x in T that minimize the sum $a + b + c$ of the distances a, b, c of x from the three sides of T.



37. [3] Let F_n denote the nth Fibonacci number. Let p be a prime not equal to 5. Show that either F_{p-1} or F_{p+1} is divisible by p. Which?

38. Fix an integer $n > 0$. Let $f(n)$ be the most number of rectangles into which a square can be divided so that every line which is parallel to one of the sides of the square intersects the interiors of at most n of the rectangles. For instance, in the following figure there are five rectangles, and every horizontal or vertical line intersects the interior of at most 3 of them. This is not best possible, since we can obviously do the same with nine rectangles.



   (a) [2] It is obvious that $f(n) \geq n^2$. Show that in fact $f(n) > n^2$ for $n \geq 3$.

   (b) [3] Show that $f(n) < \infty$, i.e., for any fixed n we cannot divide a square into *arbitrarily many* rectangles with the desired property. In fact, one can show $f(n) \leq n^n$.

   (c) [5] Find the actual value of $f(n)$. The best lower bound known is $f(n) \geq 3 \cdot\ 2^{n-1}\ -\ 2$.