SOME HINTS AND ANSWERS
TO 18.S34 SUPPLEMENTARY PROBLEMS

(Fall 2007)

2. (b) *Answer:* $(n^3 + 3n^2 + 8n)/6$, which is 13 for $n = 3$. For a picture, see M. Gardner, *The 2nd Scientific American Book of Mathematical Puzzles & Diversions*, Simon and Schuster, New York, 1961, p. 150.

   (c) 18, according to the previous reference, p. 149.

4. (b) Hint: Let $f(n)$ be the last nonzero digit of $n!$, so $f(1) = 1,\ f(2) = 2,\ f(3) = 6,\ f(4) = 4,\ f(5) = f(6) = 2$, etc. Use the identity

$$(5n)! = 10^n n! \prod_{i=0}^{n-1} \frac{(5i+1)(5i+2)(5i+3)(5i+4)}{2}$$

to show that $f(5n) \equiv 2^n f(n) \pmod{10}$.

Note: The complete answer for evaluating $f(n)$ is the following. Let $\cdots a_2 a_1 a_0$ be the base 5 expansion of n (so $n = \sum a_i 5^i,\ 0 \le a_i \le 4$). Then if $n \ne 3$, we have

$$f(n) \equiv 2^{a_1 + 2a_2 + 3a_3 + \cdots + |\{i\,:\,a_i = 2\}| + 2\cdot|\{i\,:\,a_i=4\}|} \pmod{10}.$$

For instance, 10000 in base 5 is 310000, so

$$f(10000) \equiv 2^{4\cdot 1 + 5\cdot 3 + 0 + 0} \equiv 2^{19} \equiv 8 \pmod{10}.$$

Hence $f(10000) = 8$.

7. (a) *Answer:* $\binom{n}{k}$ is odd if and only if the following holds: If $n = a_0 + a_1 2^1 + a_2 2^2 + \cdots$ and $k = b_0 + b_1 2^1 + b_2 2^2 + \cdots$ denote the binary expansions of n and k, then $b_i \le a_i$ for all i (i.e., $b_i = 0$ if $a_i = 0$). This is a result of Lucas.

   (b) *Answer:* The largest power of p dividing $\binom{n}{k}$ is equal to the number of *carries* in adding k and $p - k$ in base p (using the usual grade-school algorithm for addition). This is a result of Kummer.

8. (a) HINT: Suppose P is a convex polygon in the plane with n sides and all angles equal. Then the side lengths $a_0, a_1, \ldots, a_{n-1}$ (in that order) are possible if and only if

$$a_0 + a_1\zeta + a_2\zeta^2 + \cdots + a_{n-1}\zeta^{n-1} = 0,$$

where $\zeta = e^{2\pi i/n}$ (a primitive nth root of unity). One also needs the fact that if $n = p$, a prime number, and if $f(x)$ is a polynomial with integer coefficients satisfying $f(\zeta) = 0$, then $f(x)$ is divisible by $1 + x + x^2 + \cdots + x^{p-1}$.

11. Here is an explicit example of such a sequence. Define for $i \geq 0$,

$$b_i = \begin{cases} 0, & \text{if the number of 1's in the binary expansion of } i \text{ is even} \\ 1, & \text{if odd.} \end{cases}$$

Thus $b_0b_1b_2\cdots = 0110100110010110\cdots$. Now define

$$a_i = \begin{cases} 1, & \text{if } b_i = b_{i+1} \\ 2, & \text{if } b_i = 0,\ b_{i+1} = 1 \\ 3, & \text{if } b_i = 1,\ b_{i+1} = 0. \end{cases}$$

Thus $a_0a_1a_2\cdots = 213231213123213\cdots$. This works!

References: This problem originated with Morse and Hedlund, and there is now a huge literature on it. Some relatively accessible references are:

- G. Braunholtz, *Amer. Math. Monthly* **70** (1963), 675–676.
- D. Hawkins and W. Mientka, *Math. Student* **24** (1956), 185–187.
- J. Leech, *Math. Gazette* **41** (1957), 277–278.
- P. A. Pleasants, *Math. Proc. Cambridge Phil. Soc.* **68** (1970), 267–274.
- J. C. Shepherdson, *Math. Gazette* **42** (1958), 306.
- I. Stewart, *Scientific American*, October, 1995, pp. 182–183.

13. False! The first counterexample is at $n = 777{,}451{,}915{,}729{,}368$. See S. W. Golomb and A. W. Hales, Hypercube Tic-Tac-Toe, in *More Games of No Chance* (R. J. Nowakowski, ed.), MSRI Publications **42**, Cambridge University Press, 2002, pp. 167–182. There it is stated that the

first counterexample is at $n = 6{,}847{,}196{,}937$, an error due to faulty multiprecision arithmetic. The correct value was found by J. Buhler in 2004 and is reported in S. Golomb, "Martin Gardner and Tictacktoe" (unpublished).

24. *Answer:* yes. The first such pair of numbers was found by R. L. Graham in 1964. At present the smallest known pair, found by M. Vsemirnov in 2004, is $(a, b) = (106276436867, 35256392432)$. See

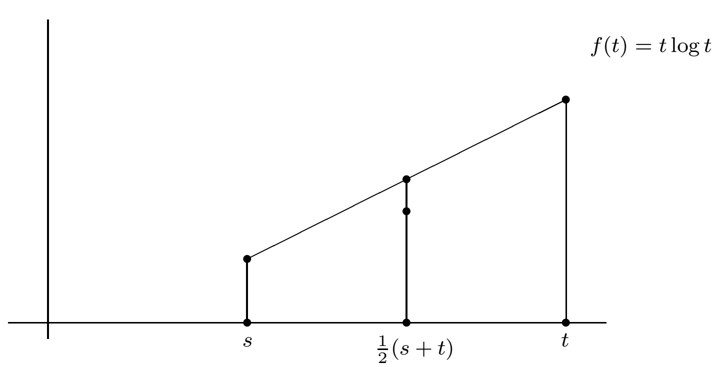
`www.cs.uwaterloo.ca/journals/JIS/VOL7/Vsemirnov/vsem5.pdf`

25. HINT:

$$\begin{aligned} a_1 &= 1 \\ a_{10} &= 16 \\ a_{100} &= 161 \\ a_{1000} &= 1618 \\ a_{10000} &= 16180 \\ a_{100000} &= 161803 \\ a_{1000000} &= 1618033. \end{aligned}$$

33. (a) The function $f(t) = t \log t$ satisfies $f''(t) = 1/t > 0$. Hence $f(t)$ is *strictly convex*, i.e., every line segment joining two points on its graph lies above the graph. Then the diagram below shows that if $0 < s < t$ then

$$f\left(\frac{1}{2}s + \frac{1}{2}t\right) < \frac{1}{2}f(s) + \frac{1}{2}f(t).$$



Now set $s = x^p$ and $t = y^p$, where $x \neq y$. Then

$$\left(\frac{x^p + y^p}{2}\right) \log \left(\frac{x^p + y^p}{2}\right) \leq \frac{1}{2} x^p \log x^p + \frac{1}{2} y^p \log y^p,$$

so

$$\log \left(\frac{x^p + y^p}{2}\right) < \frac{x^p \log x^p + y^p \log y^p}{x^p + y^p}. \tag{2}$$

Let $M(p) = M_p(x, y) = \left(\frac{x^p + y^p}{2}\right)^{1/p}$. It is easy to compute that

$$\frac{p^2 M'(p)}{M(p)} = \frac{x^p \log x^p + y^p \log y^p}{x^p + y^p} - \log \left(\frac{x^p + y^p}{2}\right).$$

Thus by (2), $p^2 M'(p)/M(p) > 0$, so $M'(p) > 0$. This means that $M(p)$ is a strictly increasing function of p, as was to be shown.

34. For a solution using only calculus (but very tricky), see Problem 58 on page 229 of G. Klambauer, *Problems and Propositions of Analysis*. This result is originally due to K. F. Gauß. See also the book J. Borwein and P. Borwein, *Pi and the AGM*, Wiley-Interscience, New York, 1998.

35. (b) *Answer:* $e^{-e} \leq x \leq e^{1/e}$. For a proof of this difficult result (originally due to L. Euler), see Problem 20 on page 186 of Klambauer's book mentioned above. For $0 < x < e^{-e}$ it is interesting to run the recurrence on a calculator and see why it doesn't converge.

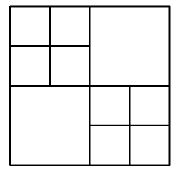
36. *All* points x in T achieve the minimum!

37. *Answer:* Let $p \neq 2, 5$. Then F_{p-1} is divisible by p if and only if the congruence

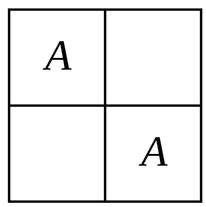
$$x^2 \equiv 5 \pmod{p}$$

has an integer solution; otherwise F_{p+1} is divisible by p. (Also F_3 is divisible by 2.) In number theory courses one shows (using the quadratic reciprocity law) that $x^2 \equiv 5 \pmod{p}$ has an integer solution (for $p \neq 2,\ 5$) if and only if $p \equiv 1$ or $p \equiv 4 \pmod{5}$.

38. (a) We have $f(3) = 10$, achieved by



More generally, if A is a partitioning of a square (meeting the conditions of the problem) for n with k squares, then the following partitioning for $n+1$ has $2k+2$ squares.



This leads easily to the lower bound $f(n) \geq 3 \cdot\ 2^{n-1}\ -\ 2$.

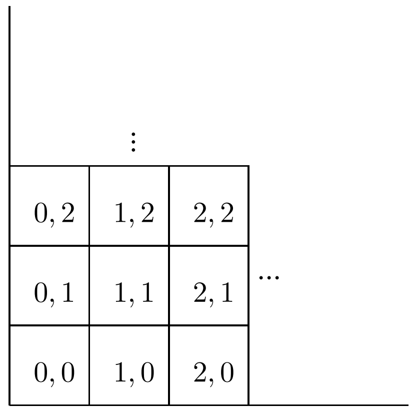
42. Simply write the numbers from 1 to n^2 in their usual order! For example,

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 \end{array} .$$

43. *Answer:* $f(n) = \binom{n}{2} + 1 = \frac{1}{2}(n^2 - n + 2)$.

44. *Answer:* Write n in binary and read it in ternary to get a_n. For instance, $1,000,000 = 2^{19} + 2^{18} + 2^{17} + 2^{16} + 2^{14} + 2^9 + 2^6$, so $a_{1,000,000} = 3^{19} + 3^{18} + 3^{17} + 3^{16} + 3^{14} + 3^9 + 3^6 = 1,726,672,221$. Once the result is guessed it is not difficult to prove by induction.

45. (b) Coordinatize the squares of the $m \times n$ rectangle as follows:



Let P be the set of coordinates of the lower left-hand squares of the $a \times b$ boards in the tiling. Let Q be the set of coordinates of the lower left-hand squares of the $b \times a$ boards. Let

$$\begin{aligned} A(x,y) &= (1+x+x^2+\cdots+x^{a-1})(1+y+y^2+\cdots+y^{b-1}) \\ B(x,y) &= (1+x+x^2+\cdots+x^{b-1})(1+y+y^2+\cdots+y^{a-1}). \end{aligned}$$

It's not hard to see from the definition of tiling that

$$\sum_{(i,j)\in P} x^i y^j A(x,y) + \sum_{(i,j)\in Q} x^i y^j B(x,y)$$

$$= (1+x+x^2+\cdots+x^{m-1})(1+y+y^2+\cdots+y^{n-1}).$$

Now let $x = y = e^{2\pi i/a}$. Then $A(x,y) = B(x,y) = 0$ [why?]. Hence

$$(1+x+x^2+\cdots+x^{m-1})(1+y+y^2+\cdots+y^{n-1}) = 0.$$

Thus either $1+x+x^2+\cdots+x^{m-1} = 0$, in which case $a \,|\, m$ [why?], or $1+y+y^2+\cdots+y^{n-1} = 0$, in which case $a \,|\, n$.

46. See the article by Stan Wagon in *American Mathematical Monthly* **94** (1987), 601–617. This is an entertaining and accessible paper which gives fourteen (!) solutions to the problem.

47. No. Consider the "inner" angle of a nonconvex quadrilateral. In a dissection of a convex polygon P into n nonconvex quadrilaterals, the sum of the angles about the inner vertex of each quadrilateral is $360°$,

for a total “inner angle sum” of at least $n \cdot 360^\circ$ (since there must be one interior vertex for each angle of a quadrilateral that is greater than 180°). But the sum of all the internal angles of a quadrilateral is 360°, so the total sum of all angles in the dissection in $n \cdot 360^\circ$. This leaves no room for angles on the boundary of P.

49. There are no such polynomials of degree less than 12. Three such polynomials (up to scalar multiplication) are known of degree 12. One is

$$13750x^{12} + 5500x^{11} - 1100x^{10} + 440x^9 - 220x^8 + 220x^7 \\ -15x^6 - 50x^5 + 10x^4 - 4x^3 + 2x^2 - 2x - 1.$$

See pp. 261–263 of M. Kreuzer and L. Robbiano, *Computational Commutative Algebra 1*, Springer-Verlag, Berlin, 2000. More generally, let $K(f(x))$ denote the number of nonzero coefficients of $f(x)$ and choose any $\epsilon > 0$. One can use the above example to construct a polynomial $g(x)$ such that $K(g(x)^2) < \epsilon K(g(x))$.