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2.004 Dynamics and Control II
Spring 2008

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING

2.004 *Dynamics and Control II*
Spring Term 2008

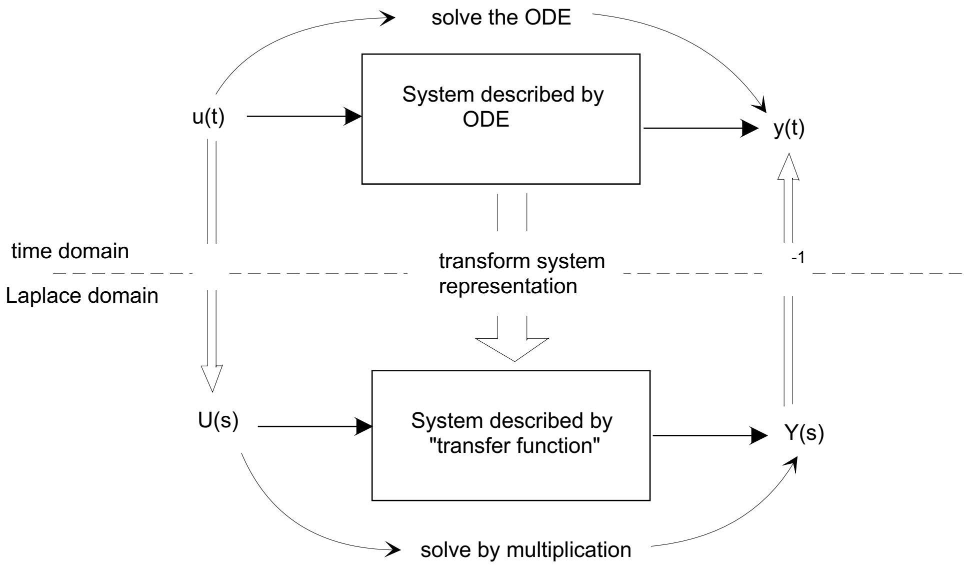
Lecture 3[1]

Reading:

- Nise: Secs. 2.2 and 2.3 (pp. 32 - 45)

1 The Laplace Transform

The reason that the Laplace transform is useful to us in 2.004 that it allows algebraic manipulation of ordinary differential equations
1) Solution of ODE's is "difficult", so
2) Transform the problem to a "domain" where the solution is easier.
3) Solve the problem in the new domain.
4) Perform the "inverse" transform to move the solution back to the original domain (if we need to).



Definition: The "one-sided" Laplace transform is an *integral transform* defined as

$$\boxed{F(s) = \mathcal{L}\left\{f(t)\right\} = \int_{0^-}^{\infty} f(t)e^{-st}dt}$$

It maps the function $f(t)$ to a function of the complex variable $s = \sigma + j\omega$ ($j = \sqrt{-1}$), and $F(s)$ is itself generally complex. We also write

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

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for the inverse transform and often

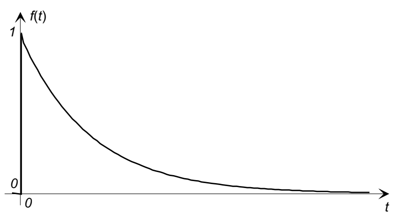
$$f(t) \overset{\mathcal{L}}{\Longleftrightarrow} F(s)$$

1.1 Some Common Examples Used in Control Theory:

(a) The unit step function: $u_s(t)$:

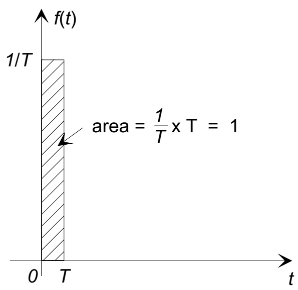
$$U_s(s) = \int_{0^-}^{\infty} 1e^{-st}dt = -\frac{1}{s}[e^{-st}]_{0^-}^{\infty} = \frac{1}{s}$$

(b) The one-sided exponential $f(t) = u_s(t)e^{-at}$ $(a > 0)$



$$F(s) = \int_{0^-}^{\infty} e^{-at}e^{-st}dt = \frac{1}{s+a}$$

(c) A very brief with unit area:



$$F(s) \approx \frac{1}{T}\int_{0^-}^{T} 1dt = 1$$

As $T \to 0$, the amplitude becomes very large and we define the Dirac delta (or impulse) function $\delta(t)$ as:

- $\delta(t) = 0$ for all $t \neq 0$
- $\delta(t)$ is undefined (infinite) at $t = 0$
- $\int_{-\infty}^{\infty} \delta(t) dt = 1$ (unit area).

and $\mathcal{L}\{\delta(t)\} = 1$ from above.

1.2 The Inverse Laplace Transform

If $F(s) = \mathcal{L}\{f(t)\}$, then $f(t) = \mathcal{L}^{-1}\{F(s)\}$ where $\mathcal{L}^{-1}$ denotes the inverse transform. In general $L^{-1}\{\}$ requires *integration along a contour* in the complex $s = \sigma + j\omega$ plane, parallel to the imaginary axis. This is rarely done in practice.

Instead, break up $F(s)$ into a sum of functions with known $\mathcal{L}^{-1}\{\}$, and use *table lookup*, for example:

■ Example 1

Find the inverse Laplace transform of

$$F(s) = \frac{4}{s^2 + 5s + 6}.$$

$$F(s) = \frac{4}{s^2 + 5s + 6} = \frac{4}{(s+3)(s+2)} = \frac{4}{s+2} - \frac{4}{s+3} \qquad \text{(partial fractions)}$$

and since $\mathcal{L}e^{-at} = \frac{1}{s+a}$, we recognize

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{4}{s+2}\right\} - \mathcal{L}^{-1}\left\{\frac{4}{s+3}\right\} = 4e^{-2t} - 4e^{-3t}$$

Note: See Nise for treatment of repeated roots

1.3 Properties of the Laplace transfrom:

We will discuss only the major properties that are useful in 2.004:

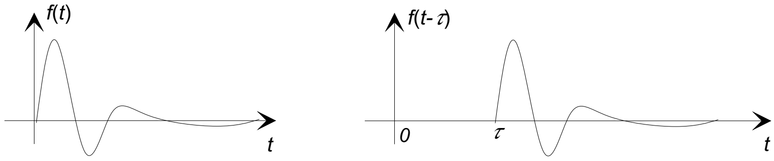
(a) Linearity: If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$ then

$$L\{af(t) + bg(t)\} = aF(s) + bG(s)$$

where a and b are constants
The linearity property is fundamental to our treatment of ODE's and linear systems.

(b) Time Shift: If $F(s) = \mathcal{L}\{f(t)\}$ then

$$\boxed{L\{f(t-\tau)\} = e^{-s\tau}F(s)}$$



This is an important property in control theory because pure delays affect system stability under feedback control.

(c) Differentiation Property: If $F(s) = \mathcal{L}\{f(t)\}$, then

$$\mathcal{L}\{\frac{df}{dt}\} = sF(s) - f(o^-)$$

$$\mathcal{L}\{\frac{d^2f}{dt^2}\} = s^2F(s) - sf(o^-) - \dot{f}(0)$$

and

$$\boxed{\mathcal{L}\{\frac{d^nf}{dt^n}\} = s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{(k-1)}(0)}$$

This is perhaps the most important property in this course.

(d) Integration property: If $F(s) = \mathcal{L}\{f(t)\}$,

$$\boxed{\mathcal{L}\{\int_0^t f(\sigma)d\sigma\} = \frac{1}{s}F(s)}$$

(e) The Final Value Theorem: If $F(s) = \mathcal{L}\{f(t)\}$, then

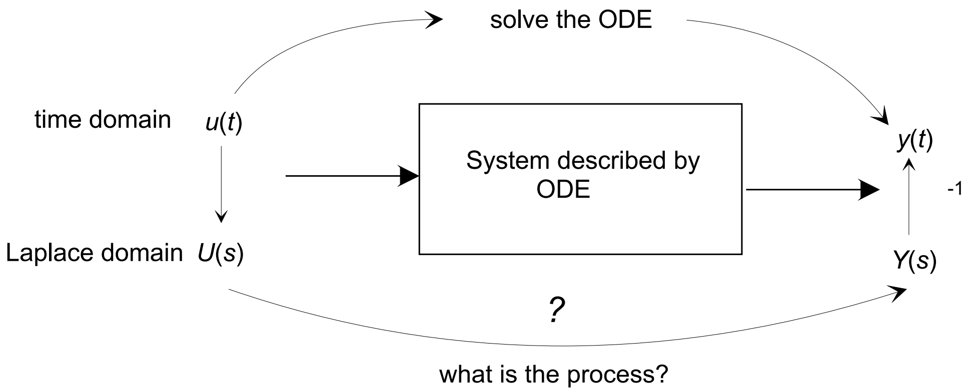
$$\boxed{\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)}$$

provided the limit exists. The f.v. theorem is useful for determining the steady-state response of systems.

2 Laplace Domain System Representation

Suppose that through modeling we have found that a system is described by a differential equation

$$a_n\frac{d^ny}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_1\frac{dy}{dt} + a_0y = b_m\frac{d^mu}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \ldots + b_1\frac{du}{dt} + b_0u$$



Assume that the system is "at rest" at time $t = 0$, that is $y(0) = 0$, $\dot{y}(0) = 0$, etc. and that $u(0^-) = 0$, $\dot{u}(0^-) = 0$, etc then using the differentiation property of the Laplace transform on each term in the ODE gives:

$$a_n s^n Y(s) + a_{n-1}s^{n-1}Y(s) + \ldots + a_0 Y(s) = b_m s^m U(s) + b_{m-1}s^{m-1}U(s) + \ldots + b_0 U(s)$$

$$[a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0]Y(s) = [b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0]U(s)$$

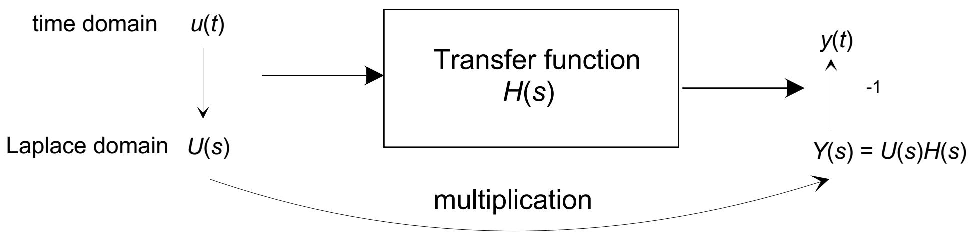
and solving for $Y(s)$

$$\boxed{Y(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0}U(s) = H(s)U(s)}$$

where H(s) is defined as the system **transfer function.**

$$H(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0} = \frac{N(s)}{D(s)}$$

where numerator coefficients come from the RHS of the ODE and the denominator coefficients come from the LHS. $H(s)$ is a rational fraction for most linear systems.



The Laplace transform (transfer function) has changed the system representation to from an ODE to an algebraic representation with a multiplicative input/output relationship.

In system dynamics and control work we use the transfer function as the primary system representation.

■ Example 2

Find the transfer function of a system represented by the ODE:

$$5\frac{d^3y}{dt^3} + 17\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 5y = 8\frac{du}{dt} + 6u$$

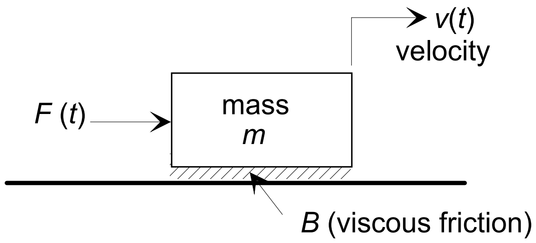
Answer:

$$H(s) = \frac{8s + 6}{5s^3 + 17s^2 + 6s + 5}$$

Note: A fundamental assumption when using the transfer function to compute responses is that the system is "at rest" at time $t = 0$.

■ Example 3

Find the response $V(s)$ of the velocity of the mass element shown below to a unit step the applied force $F(t)$



$$m\frac{dv}{dt} + Bv = F(t)$$

From the differential equation

$$(ms + B)V(s) = F(s)$$

$$V(s) = \frac{1}{ms + B}F(s).$$

For a unit-step in the force $F(t)$, $F(s) = 1/s$ and

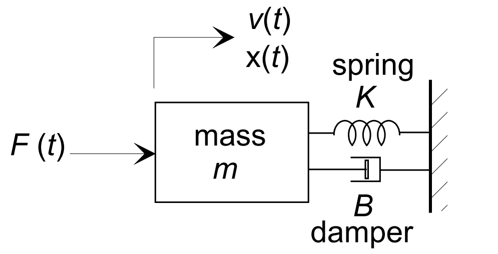
$$V(s) = \frac{1}{ms + B} \times \frac{1}{s} = \frac{B}{s} - \frac{B}{s + B/m}$$

Taking the inverse Laplace transform gives the response

$$v(t) = \mathcal{L}^{-1}\{V(s)\} = \frac{1}{B}\left(1 - e^{-\frac{B}{m}t}\right)$$

■ Example 4

Find the transfer function relating: a) $x(t)$, b) $v(t)$ to $F(t)$ for the system



a) From a force balance

$$m\ddot{x} + B\dot{x} + Kx = F.$$

By inspection

$$H(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + Bs + k}$$

b) Since $v(t) = dx/dt$,

$$m\dot{v} + Bv + K\int_0^t vdt = F$$

or

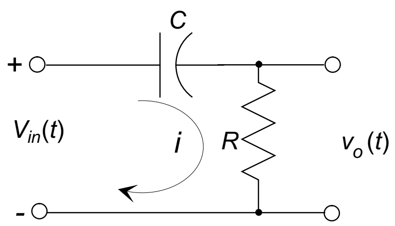
$$m\ddot{v} + B\dot{v} + Kv = \dot{F}$$

and

$$H(s) = \frac{V(s)}{F(s)} = \frac{s}{ms^2 + Bs + K}$$

■ Example 5

Find the transfer function of the electrical circuit



What we know:

(1)

$$V_{in}(t) = v_c + v_r \qquad \text{(KVL)}$$

(2)

$$v_o = v_R = \frac{1}{R} i_R$$

(3)

$$i_R = i_c \qquad \text{(KCL)}$$

From (1):

$$V_{in}(t) = v_c + v_R = \frac{1}{C}\int_0^t i_c dt + v_R$$

Differentiate and use (2):

$$\dot{V}_{in}(t) = \frac{1}{C} i_C + \dot{v}_R = \frac{1}{RC} v_R + \dot{v}_R$$

Use $v_o = v_R$ to obtain:

$$\dot{V}_{in}(t) = \frac{1}{RC} v_o + \dot{v}_o.$$

Take the Laplace transform of both sides, and use the derivative property to give

$$H(s) = \frac{V_o(s)}{V_{in}(s)} = \frac{RCs}{RCs + 1}$$
