2.004 Dynamics and Control II
Spring 2008

Massachusetts Institute of Technology

Department of Mechanical Engineering

2.004 *Dynamics and Control II*
Spring Term 2008

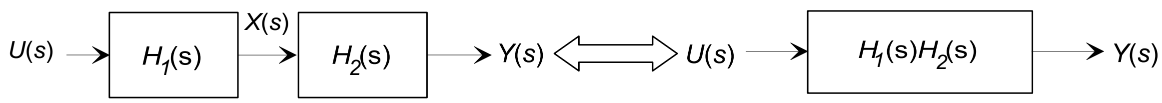
Lecture 4[1]

Reading:

- Nise: Secs. 5.1–5.3

1 Block Diagram Algebra (Interconnection Rules)

a)Series (Cascade) Connection:

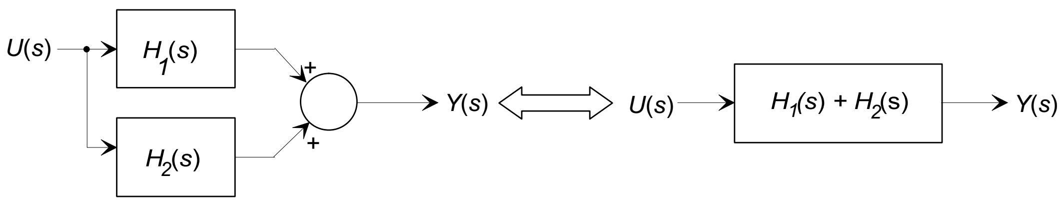


Since the output of the first block is $X(s) = H_1(s)U(s)$,

$$Y(s) = H_2(s)X(s) = H_1(s)H_2(s)U(s)$$

Note: This is only true if the connection of $H_s(s)$ to $H_1(s)$ does not alter the output of $H_1(s)$ – known as the "non-loading" condition.

b)Parallel Connection In this case the input $U(s)$ is applied to both inputs and the outputs are summed:



$$Y(s) = H_1(s)U(s) + H_2(s)U(s) = (H_1(s) + H_2(s))U(s)$$

■ Example 1

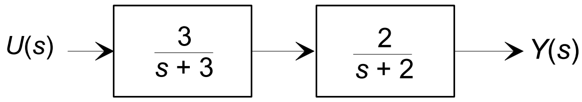
Express

$$H(s) = \frac{6}{s^2 + 5s + 6}$$

as (a) a series connection, and (b) a parallel connection of first-order blocks

a) <u>Series:</u>

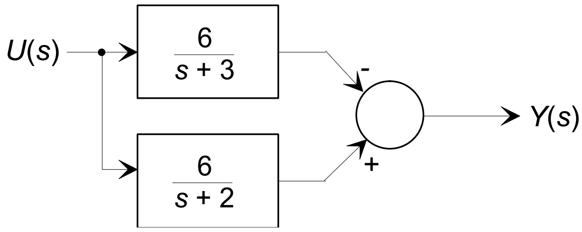
$$H(s) = \frac{6}{s^2+5s+6} = \frac{3}{s+3} \times \frac{2}{s+2}$$



order of blocks is arbitrary

b) <u>Parallel:</u> Using partial fractions we find

$$H(s) = \frac{6}{s+2} - \frac{6}{s+3}$$



<u>Notes:</u> a) These two systems are equivalent. b) A partial fraction expansion is effectively a parallel implementation. c) A factored representation of $H(s)$ is effectively a series implementation.

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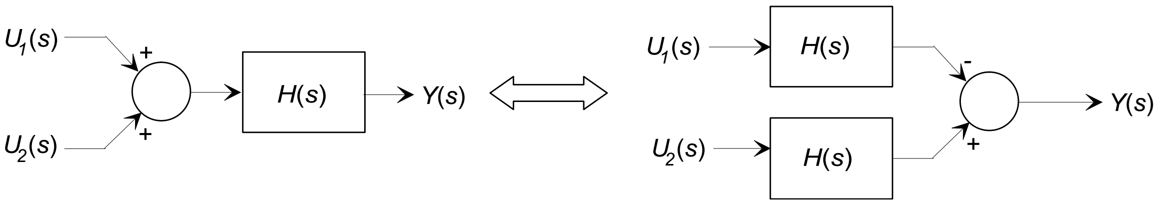
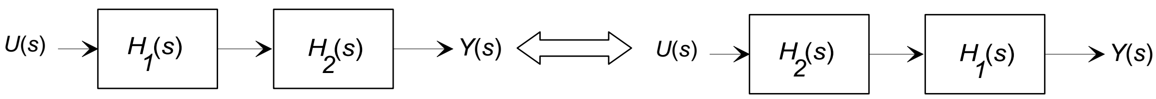
c)Associative Rule:



$$Y(s) = (U_1(s) + U_2(s))H(s) \equiv U_1(s)H(s) + U_2(s)H(s)$$

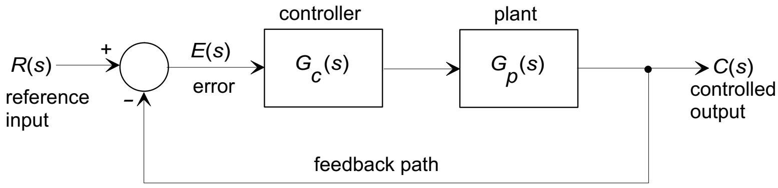
d)Commutative Rule:



The order does not matter in a series connection.

2 The "Closed-Loop" Transfer Function

a) Unity feedback



Notes:

(a) The term *unity feedback* means that the *actual* output value is used to generate the error signal (the feedback gain is 1).

(b) In control theory transfer functions in the "forward" path are often designated by $G(s)$ (see below).

(c) It is common to use $R(s)$ to designate the *reference* (desired) input, and C(s) to designate the *controlled* (output) variable.

From the block diagram:

$$C(s) = (G_p(s)G_c(s))E(s)$$

and

$$E(s) = R(s) - C(s)$$

or

$$C(s) = G_p(s)G_c(s)(R(s) - C(s)$$

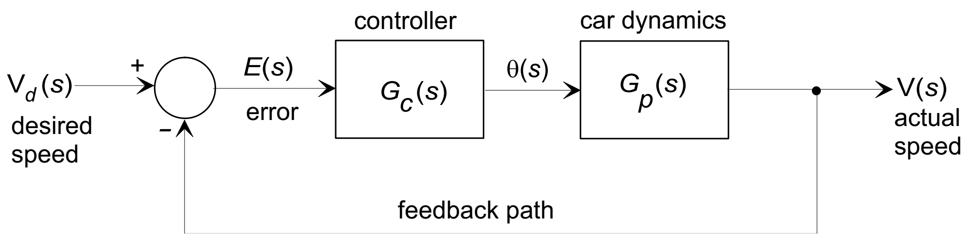
Rearranging:

$$\boxed{G_{cl}(s) = \frac{C(s)}{R(s)} = \frac{G_c(s)G_ps}{1 + G_c(s)G_ps}}$$

is the unity feedback closed-loop transfer function.

■ Example 2

Find the closed-loop transfer function for the automobile cruise control example:



For the car $m\dot{v} + Bv = F_p = K_e\theta$ so that

$$G_p(s) = \frac{V(s)}{\theta(s)} = \frac{K_s}{ms + B}$$

For the controller $\theta(s) = K_c E(s)$

$$G_c(s) = \frac{\theta(s)}{E(s)} = K_c$$

Then from above

$$G_{cl}(s) = \frac{V(s)}{V_d(s)} = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)}$$

$$G_{cl}(s) = \frac{\frac{K_cK_e}{ms+B}}{1 + \frac{K_cK_e}{ms+B}} = \frac{K_cK_e}{ms + (B + K_cK_e)},$$

and by inspection the closed-loop differential equation is

$$m\dot{v} + (B + K_cK_e)v = K_cK_e v_d.$$

Aside:

Use the Laplace transform final value theorem to find the steady state velocity to a step input $v_d(t) = v_d$

For the step input

$$v_d(s) = \frac{v_d}{s}$$

and in the Laplace domain

$$v(s) = G_{c1}(s)V_d(s) = \frac{K_cK_e}{ms + (B + K_cK_e)}\frac{v_d}{s}$$

The F.V. theorem states $lim_{t\to\infty} f(t) = lim_{s\to 0} sF(s)$ so that

$$v_{ss} = lim_{t\to\infty} v(t) = lim_{s\to 0} s\frac{K_cK_e}{ms + (B + K_cK_e)}\frac{v_d}{s}$$

$$v_s s = \frac{K_cK_e}{B + K_cK_e}$$

which is same as we obtained before.

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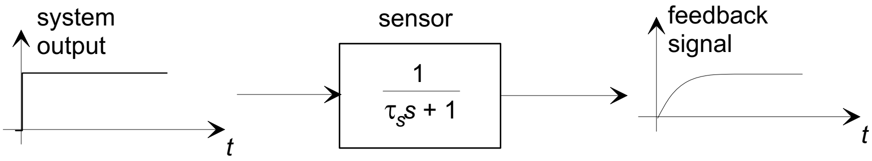
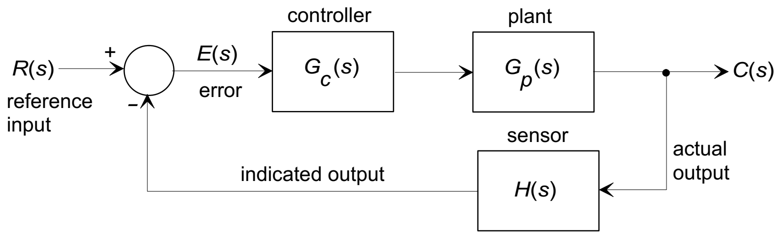
3 Closed-Loop Transfer Function With Sensor Dynamics:

Until now we have assumed that the output variable $y(t)$ is measured instantaneously, and without error. Frequently the sensor has its own dynamics - for example the sensor might be temperature measuring device modeled as a first-order system:



where τ_s is the sensor time constant.

The closed-loop block diagram is



where $H(s)$ is the transfer function of the sensor. In this case:

$$C(s) = (G_c(s)G_p(s))E(s)$$

but now $E(s)$ is the *indicated* error (as opposed to the actual error):

$$E(s) = R(s) - H(s)C(s)$$

so

$$C(s) = G_c(s)G_p(s)(R(s) - H(s)C(s))$$

or

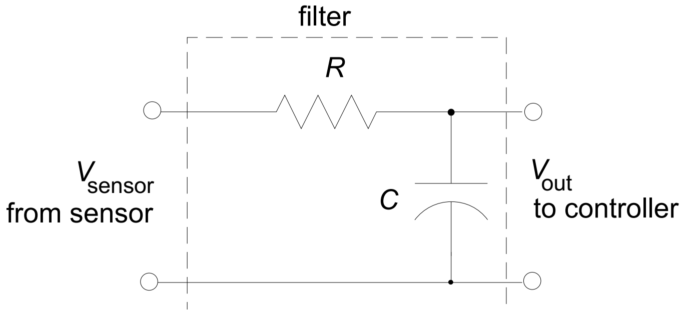
$$C(s)(1 + G_e(s)G_p(s)H(s)) = G_e(s)G_p(s)H(s).$$

$$\boxed{G_{cl}(s) = \frac{C(s)}{R(s)} = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)H(s)}}$$

is the modified closed-loop transfer function.

■ Example 3

Suppose that velocity sensor in the cruise control is "noisy", and a simple electrical filter is used to *smooth the output.* Find the effect of the filter on the closed-loop dynamics.

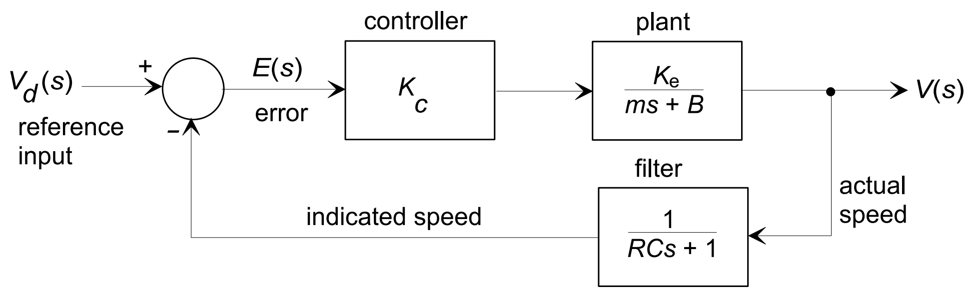


Using Kirchoff's Voltage Law (KVL) we find

$$RC\dot{v}_{out} + v_{out} = v_{sensor}$$

so that

$$H(s) = \frac{V_{out}(s)}{V_{sensor}(s)} = \frac{1}{RCs + 1}$$



Then the closed-loop transfer function is

$$\begin{aligned} G_{cl}(s) &= \frac{V(s)}{V_a(s)} = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)H(s)} = \frac{\frac{K_cK_e}{ms+B}}{1 + \frac{K_cK_e}{(ms+B)(RCs+1)}} \\ &= \frac{K_cK_e(RCs+1)}{(ms+B)(RSs+1) + K_cK_e} \end{aligned}$$

$$\boxed{G_{cl}(s) = \frac{K_cK_e(RCs+1)}{mRCs^2 + (BRC + m)s + (B + K_cK_e)}}$$

and the differential equation relating the speed of the car to the desired speed command is now

$$mRC\ddot{v} + (BRC + m)\dot{v} + (B + K_cK_e)\dot{v} = K_cK_eRC\dot{v}_d + K_cK_ev_d$$

and we note:
1) we now have a second-order system - the dynamics may change significantly,
2) we have derivative action on the RHS of the differential equation.
