2.004 Dynamics and Control II
Spring 2008

Massachusetts Institute of Technology
Department of Mechanical Engineering

2.004 *Dynamics and Control II*
Spring Term 2008

Lecture 6[1]

Reading:

- Nise: Sec. 2.4 (pages 45–55)
- Class Handout: *Modeling Part 1: Energy and Power Flow in Linear Systems*
  Sec. 1 (Introduction)
  Sec. 4 (Electrical System Elements)

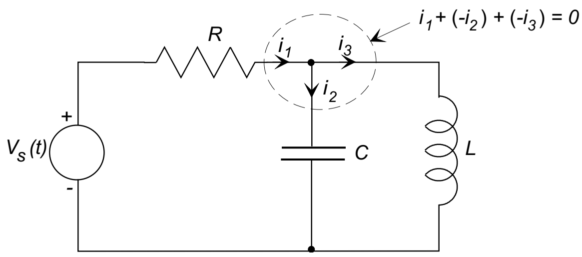
1 Modeling Electrical Systems (continued)

In Lecture 5 we examined the primitive electrical elements (capacitors inductors and resistors), and sources (voltage source and current source). We now look at how these elements behave when connected together in a circuit.

Interconnection Laws:

(a) Kirchoff's Current Law (KCL): The sum of currents flowing *into*(or out of) a junction is zero. In the figure below, at the circled junction we sum the currents into the junction to find

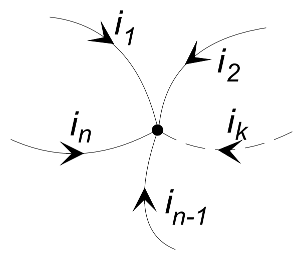
$$i_1 - i_2 - i_3 = 0$$



We will define a junction as a *node*, and if there are n circuit branches attached to a node

$$\sum_{i=1}^{n} i_n = 0$$

where we define the convention that positive current flow is into the node.

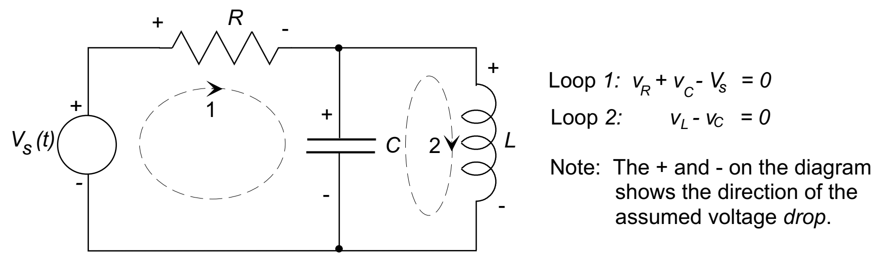


Kirchoff's Voltage Law (KVL): The sum of voltage *drops* around any closed loop in a circuit is zero. The assumed sign convention for the voltage drop on each element must be defined. Two clockwise loops are shown in the figure below. For loop (1)

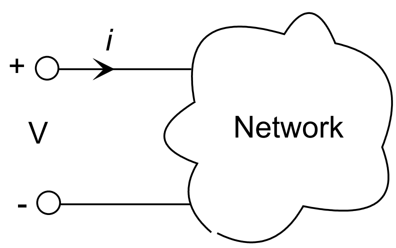
$$v_R + v_C - V_s(t),$$

while for loop (2)

$$v_L - v_C(t) = 0.$$



Electrical Impedance:



Define the *impedance* of an element or passive circuit as a *transfer function* relating current $I(s)$ to voltage $V(s)$ at its terminals:

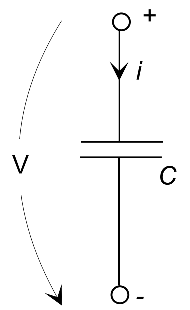
$$\boxed{Z(s) = \frac{V(s)}{I(s)}}$$

In addition we can define the *admittance* $Y(s)$ as the reciprocal of the impedance:

$$\boxed{Y(s) = \frac{1}{Z(s)} = \frac{I(s)}{V(s)}}$$

The Impedance of Passive Electrical Elements

(a) The Capacitor:



For the capacitor

$$i = C\frac{dv}{dt}.$$

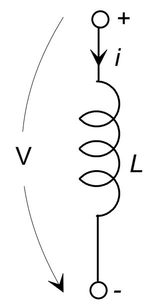
Taking the Laplace Transform:

$$I(s) = CsV(s)$$

$$\boxed{Z_C(s) = \frac{V(s)}{I(s)} = \frac{1}{sC}}$$

or the admittance $Y_C(s) = sC$.

(b) The Inductor:



For the inductor

$$v = L\frac{di}{dt}.$$

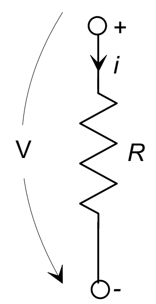
Taking the Laplace Transform:

$$V(s) = LsI(s)$$

$$\boxed{Z_L(s) = \frac{V(s)}{I(s)} = sL}$$

or the admittance $Y_L(s) = 1/sL$.

(c) The Resistor:



For the resistor

$$v = Ri.$$

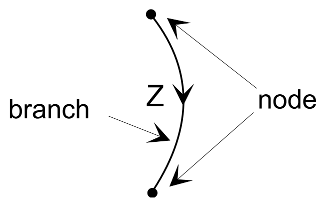
Taking the Laplace transform

$$V(s) = RI(s)$$

$$\boxed{Z_R(s) = \frac{V(s)}{I(s)} = R}$$

or the admittance $Y_R = 1/R$.

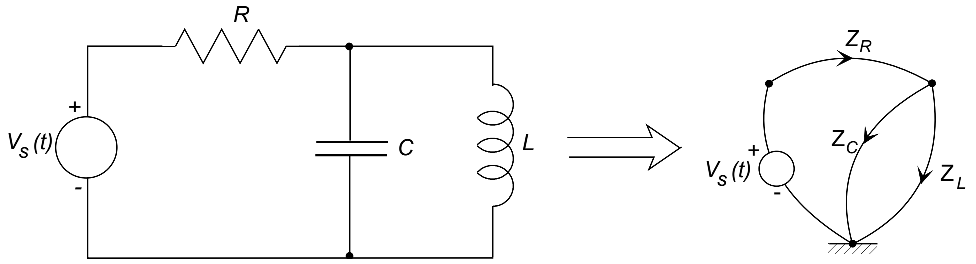
Impedance Nomenclature: We now introduce a graphical representation that will be used to denote systems in many energy domains.



The impedance is drawn as a graph *branch* between two *nodes*. Nodes represent junctions between the elements in the circuit. The arrow on the branch indicates both the assumed direction of voltage drop across the element, and the assumed current direction.

■ Example 1

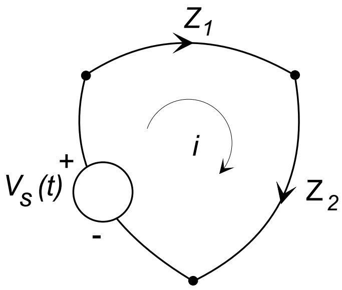
The electrical circuit, consisting of a capacitor C, an inductor L, and a resistor R is shown below:



The impedance graph is shown on the right. The nodes on the graph represent *points of distinct voltage* in the circuit.

Impedance Connection Rules

(a) Series connection: Two or more elements are defined to be connected in series *if they share a common current.* For the two elements Z_1 and Z_2 in series below:



Using KCL at the junction between Z_1 and Z_2:

$$i_{Z_1} = i_{Z_2} = i$$

Using KVL around the loop: $v_{Z_1} + v_{Z_2} - V_s = 0$

$$V_s = iZ_1 + iZ_2$$

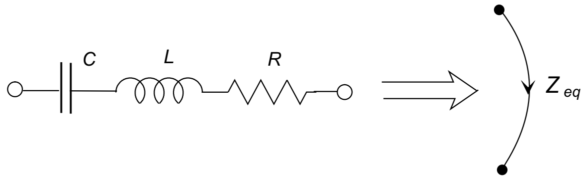
$$\boxed{Z_{eq} = \frac{V(s)}{I(s)} = Z_1 + Z_2}$$

In general with n impedances Z_i $(i = 1, \ldots, n)$ in series:

$$\boxed{Z_{eq} = \sum_{i=1}^{n} Z_i}$$

■ Example 2

For the tree elements in series below:

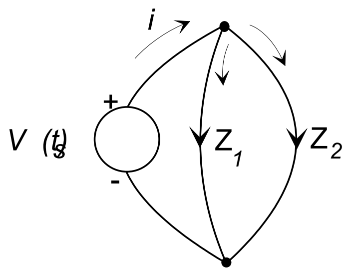


$$Z_{eq} = Z_C + Z_L + Z_R = \frac{1}{sC} + sL + R$$

or expressing the impedance as a transfer function (a ratio of polynomials):

$$Z_{eq} = \frac{V(s)}{I(s)} = \frac{LCs^2 + RCs + 1}{Cs}$$

(b) Parallel connection: Two or more elements are defined to be connected in parallel if they *share a common voltage*. For the two elements Z_1 and Z_2 in parallel below:



Using KVL:

$$v_{Z_1} = v_{Z_2} = V_s$$

Using KCL at the node:

$$i_s = i_{Z_1} + i_{Z_2}$$

$$\frac{1}{Z_{eq}} = \frac{I}{V} = \frac{i_{Z_1} + i_{Z_2}}{V}$$

$$\boxed{\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2}}$$

In general for n impedances Z_i $(i = 1, \ldots, n)$ in parallel, the equivalent impedance is:

$$\boxed{\frac{1}{Z_{eq}} = \sum_{i=1}^{n} \frac{1}{Z_i}}$$

Alternatively, using admittances $Y = 1/Z$

$$y_{eq} = \frac{1}{Z_{eq}} = \sum_{i=1}^{n} Y_i.$$

<u>Note:</u> For $N = 2$ we can write

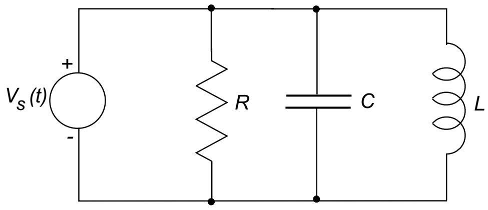
$$\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} = \frac{Z_1 + Z_2}{Z_1 Z_2}$$

which leads to the very common representation

$$\boxed{Z_{eq} = \frac{Z_1 Z_2}{Z_1 + Z_2}}$$

■ Example 3

Find the impedance of a capacitor C, and inductor L and a resistor R connected in parallel:

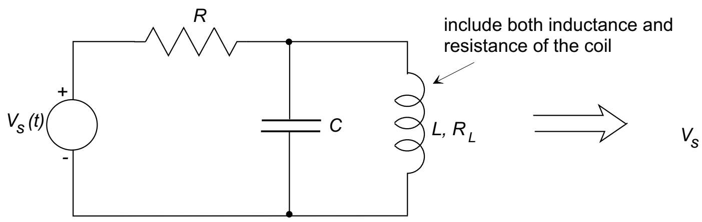


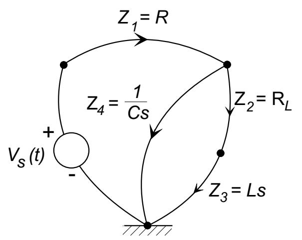
$$\begin{aligned} \frac{1}{Z} &= \frac{1}{1/sC} + \frac{1}{sL} + \frac{1}{R} \\ &= sC + \frac{1}{sL} + \frac{1}{R} \\ &= \frac{LCRs^2 + Ls + R}{RLs} \end{aligned}$$

$$\boxed{Z = \frac{V(s)}{I(s)} = \frac{RLs}{LCRs^2 + Ls + R}}$$

■ Example 4

Find the impedance of the following circuit, assuming we should include resistance and inductance of the coil:

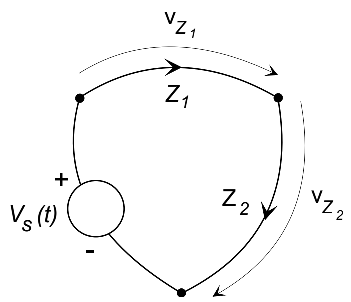




$$\begin{aligned} Z &= Z_1 + Z_4 \parallel (Z_2 + Z_3) \\ &= Z_1 + \frac{Z_4(Z_2 + Z_3)}{Z_4 + Z_2 + Z_3} \\ &= R + \frac{(1/sC)(R_L + Ls)}{1/sC + R_L + Ls} \\ &= R + \frac{R_L + Ls}{LCs^2 + R_LCs + 1} \end{aligned}$$

$$Z = \frac{V(s)}{I(s)} = \frac{RLCs^2 + (RR_LC + L)s + (R + R_L)}{LCs^2 + R_LCs + 1}$$

The Voltage Divider: Consider two impedances in series with voltage V across them:



$$I(s) = \frac{V(s)}{(Z_1 + Z_2)}$$

and

$$V_{Z_2}(s) = I(s)Z_2 = V_{Z_2}(s) = \frac{Z_2}{Z_1 + Z_2}V(s).$$

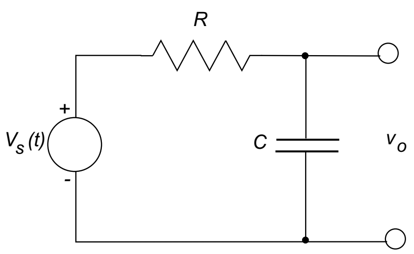
Similarly

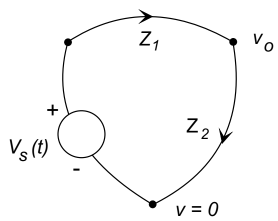
$$V_{Z_1}(s) = \frac{Z_1}{Z_1 + Z_2}V(s).$$

The voltage divider relationship may be used to find the transfer function of many simple systems.

■ Example 5

Find the transfer function relating V_0 to V_s in the following circuit





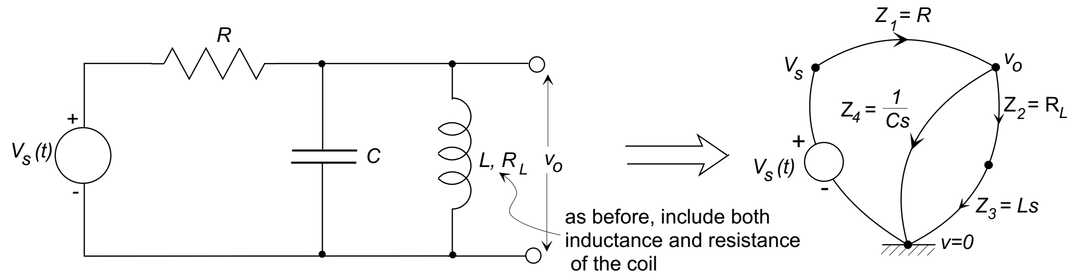
Use the voltage divider relationship

$$V_0 = \frac{Z_2}{Z_1 + Z_2}V_s = \frac{1/sC}{R + 1/sC}V_s$$

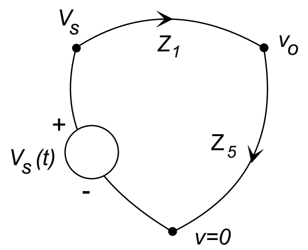
$$H(s) = \frac{V_0(s)}{V(s)} = \frac{1}{RCs + 1}$$

■ Example 6

Find the transfer function relating V_0 to V_s in the following circuit:



Reduce the impedance graph to a series connection of two elements



$$V_0 = \frac{Z_5}{Z_1 + Z_5} V_s$$

where

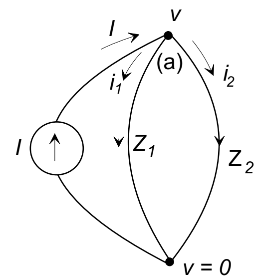
$$\begin{aligned} Z_5 &= Z_4 \parallel (Z_2 + Z_3) = \frac{Z_4(Z_2 + Z_3)}{Z_4 + Z_2 + Z_3} \\ &= \frac{(1/sC)(R_L + L_s)}{1/sC + R_L + Ls} \end{aligned}$$

Using the voltage divider relationship, the transfer function is

$$H(s) = \frac{V_0(s)}{V_s(s)} = \frac{Z_5}{Z_1 + Z_5} = \frac{\frac{R_L + Ls}{LCs^2 + R_L Cs + 1}}{R_1 + \frac{R_L + L_s}{LCs^2 + R_L Cs + 1}}$$

$$\boxed{H(s) = \frac{R_L + Ls}{R_1 LCs^2 + (R_1 R_L C + L)s + (R_1 + R_L)}}$$

The Current Divider: Consider two impedances in parallel:



Using KCL at the top node (a),

$$I - i_1 - i_2 = 0 \qquad \text{or} \qquad i_1 + i_1 = I$$

But $i_1 = V/Z_1$, and $i_2 = V/Z_2$ so that

$$\frac{V}{Z_1} + \frac{V}{Z_2} = I \qquad \text{or} \qquad V = \frac{1}{1/Z_1 + 1/Z_2} I$$

$$\boxed{i_1 = \frac{V}{Z_1} = \frac{1/Z_1}{(1/Z_1 + 1/Z_2)} I = \frac{Y_1}{Y_1 + Y_2} I}$$

Similarly

$$i_2 = \frac{Y_2}{Y_1 + Y_2} I.$$

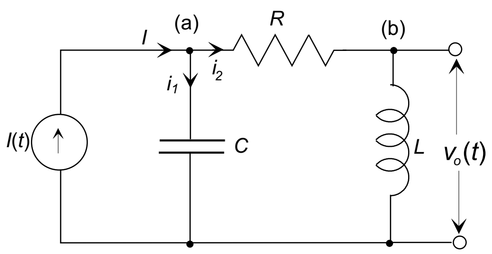
The current divider may be used to find transfer functions for some simple circuits.

■ Example 7

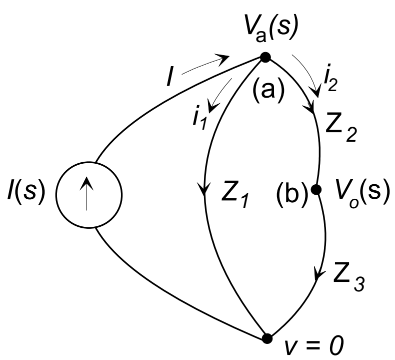
Find the transfer function

$$H(s) = \frac{V_o(s)}{I(s)}$$

in the following circuit:



Draw the system as an impedance graph:



Let $Z_1 = 1/sC$, $Z_2 = R$, and $Z_3 = sL$. We will use $V_o(s) = I_2(s)Z_3$ (at node (b)), and find $I_2(s)$ from the current division at node (a):

$$\begin{aligned} I_2(s) &= \frac{\frac{1}{Z_2+Z_3}}{\frac{1}{Z_1} + \frac{1}{Z_2+Z_3}} I(s) = \frac{1}{(1/Z_1)(Z_2 + Z_3) + 1} I(s) \\ &= \frac{1}{Cs(R + Ls) + 1} I(s) = \frac{1}{LCs^2 + RCs + 1} I(s) \end{aligned}$$

$$V_o(s) = I_2(s)Ls = \frac{Ls}{LCs^2 + RCs + 1} I(s)$$

or

$$\boxed{H(s) = \frac{V_o(s)}{I(s)} = \frac{Ls}{LCs^2 + RCs + 1}}$$
