2.004 Dynamics and Control II
Spring 2008

Massachusetts Institute of Technology
Department of Mechanical Engineering

2.004 *Dynamics and Control II*
Spring Term 2008

Lecture 7[1]

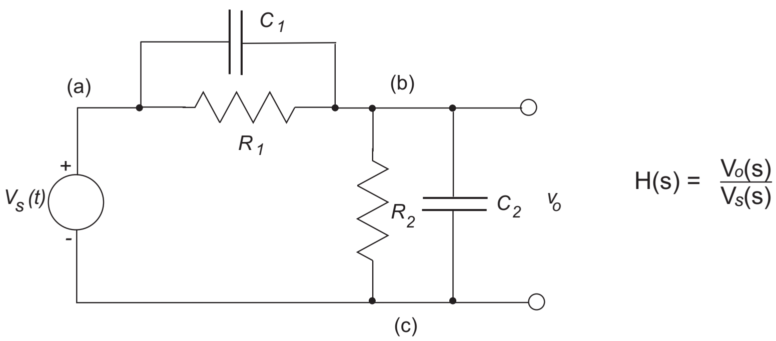
Reading:

- Nise: Sec. 2.4

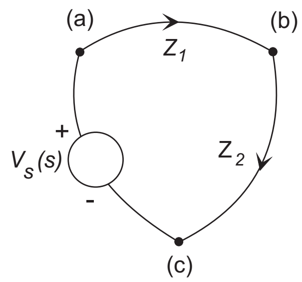
1 Transfer Function Generation by Simplification

■ Example 1

Find the transfer function for a "lead-lag" compensator



Draw as



where

$$Z_1 = \frac{1}{C_1 s} \parallel R_1 = \frac{R_1}{R_1 C_1 s + 1}$$
$$Z_2 = \frac{1}{C_2 s} \parallel R_2 = \frac{R_2}{R_2 C_2 s + 1}$$

Using the voltage divider formed by Z_1 and Z_2

$$\begin{aligned} V_o &= \frac{Z_2}{Z_1 + Z_2} V_s \\ &= \frac{\left(\frac{R_2}{R_2 C_2 s + 1}\right)}{\left(\frac{R_1}{R_1 C_1 s + 1}\right) + \left(\frac{R_2}{R_2 C_2 s + 1}\right)} V_s \\ &= \frac{R_2 R_1 C_1 s + R_2}{R_1 R_2 (C_1 + C_2) s + (R_1 + R_2)} V_s \end{aligned}$$

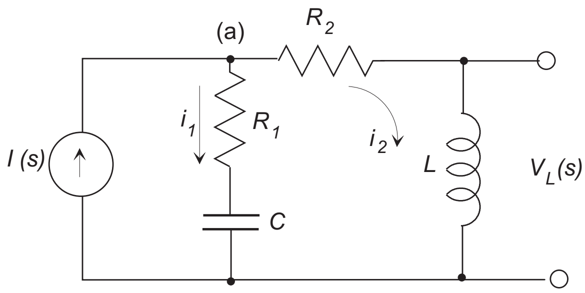
$$\boxed{H(s) = \frac{V_o(s)}{V_s(s)} = \frac{R_2 R_1 C_1 s + R_2}{R_1 R_2 (C_1 + C_2) s + (R_1 + R_2)}}$$

■ Example 2

Find the transfer function

$$H(s) = \frac{V_L(s)}{I(s)}$$

for the circuit:



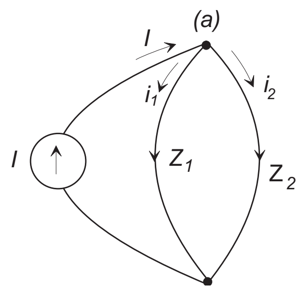
We note that

$$V_L(s) = Z_L i_2(s) = L s i_2(s)$$

Combine elements to let

$$\begin{aligned} Z_1 &= \frac{1}{Y_1} = R_1 + \frac{1}{Cs} \\ Z_2 &= \frac{1}{Y_2} = R_2 + Ls \end{aligned}$$

and use the current divider relationship at node **(a)**.



$$i_2(s) = \frac{Y_2}{Y_1 + Y_2}$$

where

$$\begin{aligned} Y_1 &= \frac{1}{Z_1} = \frac{sC}{sCR_1 + 1} \\ Y_2 &= \frac{1}{Z_2} = \frac{1}{sL + R_2}. \end{aligned}$$

Then

$$i_2(s) = \frac{R_1Cs + 1}{Cs(R_2 + Ls) + (R_1Cs + 1)} I(s)$$

and

$$V_L(s) = Lsi_2(s) = \frac{LCR_1s^2 + Ls}{LCs^2 + C(R_1 + R_2)s + 1} I(s)$$

so that the transfer function is

$$\boxed{H(s) = \frac{V_L(s)}{I(s)} = \frac{LCR_1s^2 + Ls}{LCs^2 + C(R_1 + R_2)s + 1}}$$

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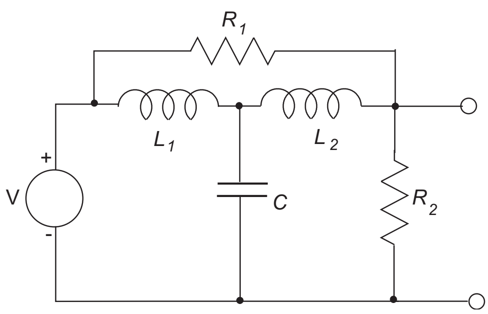
2 Transfer Function Generation through Mesh (Loop) Currents

This method expresses the system dynamics as a set of simultaneous algebraic equations in a set of internal *mesh* (or *loop*) currents. It is useful for complex circuits containing a voltage source.

The following example sets out the method in a series of steps.

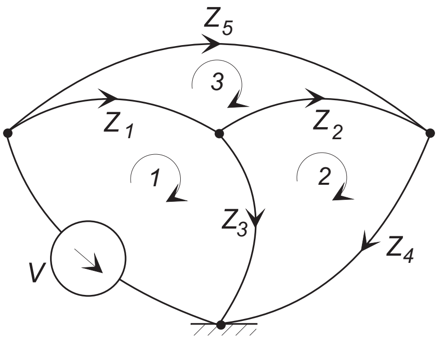
■ Example 3

Find the transfer function of a “bridged-T” filter:



Note: This is a difficult example to solve using impedance reduction methods. It is, however, well suited to the mesh current method.

Draw the system as an impedance graph:



Define a set of (clockwise) loops as shown, ensuring that every graph branch is covered by at least one loop. The loops 1, 2 and 3 are somewhat arbitrary. We assume hypothetical continuous mesh currents i_1, i_2, and i_3 that flow around each loop.

Step 1: Write loop (compatibility) equations for each loop (using the arrows on the graphs branches to define the direction of the voltage drop):

$$\begin{aligned} V_{Z_1} + V_{Z_3} - V &= 0 \\ V_{Z_2} + V_{Z_4} - V_{Z_3} &= 0 \\ V_{Z_5} + V_{Z_2} - V_{Z_1} &= 0 \end{aligned}$$

Step 2: Define the mesh currents i_1, i_2, and i_3 and write the current in each branch in terms of the mesh currents (use the arrows on the loops to define the signs):

$$\begin{aligned} i_{Z_1} &= i_1 - i_3 \\ i_{Z_2} &= i_2 - i_3 \\ i_{Z_3} &= i_1 - i_2 \\ i_{Z_4} &= i_2 \\ i_{Z_5} &= i_3 \end{aligned}$$

Step 3: Write the mesh equations in terms of the mesh currents

$$\begin{aligned} Z_1(i_1 - i_3) + Z_3(i_1 - i_2) &= V \\ Z_2(i_2 - i_3) + Z_4 i_2 - Z_3(i_1 - i_2) &= 0 \\ Z_5 i_3 - Z_2(i_2 - i_3) - Z_1(i_1 - i_3) &= 0 \end{aligned}$$

which is a set of 3 simultaneous algebraic equations in the loop currents i_1, i_2, and i_3.

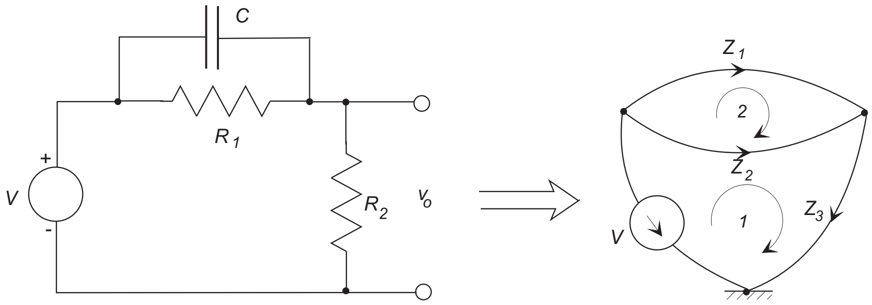
Step 4: We note that the output is $V_{Z_4} = i_2 Z_4$, therefore solve for i_2 and create the transfer function in terms of the impedances.

■ Example 4

Use the mesh current method to find the transfer function

$$H(s) = \frac{V_o(s)}{V(s)}$$

in the "lead" network:



Let $Z_1 = 1/Cs$, $Z_2 = R_1$, and $Z_3 = R_2$.

Step1: The loop equations are:

$$\begin{aligned} V_{Z_2} + V_{Z_3} - V &= 0 \\ V_{Z_1} - V_{Z_2} &= 0 \end{aligned}$$

Step 2: The mesh currents are i_1 and i_2, and

$$\begin{aligned} i_{Z_1} &= i_2 \\ i_{Z_2} &= i_1 - i_2 \\ i_{Z_3} &= i_1 \end{aligned}$$

Step 3: Rewrite the loop equations

$$\begin{aligned} Z_2(i_1 - i_2) + Z_3 i_1 &= V \\ Z_1 i_2 - Z_2(i_1 - i_2) &= 0 \end{aligned}$$

or

$$\begin{aligned} (Z_2 + Z_3)i_1 - Z_2 i_2 &= V \\ -Z_2 i_1 + (Z_1 + Z_2)i_2 &= 0 \end{aligned}$$

In matrix form

$$\begin{bmatrix} Z_2 + Z_3 & -Z_2 \\ -Z_2 & Z_1 + Z_2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} V \\ 0 \end{bmatrix}$$

Step 4: $V_o = i_1 Z_3$, so we solve for i_1 (using any method). Using Cramer's Rule:

$$i_1 = \frac{\begin{vmatrix} V & -Z_2 \\ 0 & Z_1 + Z_2 \end{vmatrix}}{\begin{vmatrix} Z_2 + Z_3 & -Z_2 \\ -Z_2 & Z_1 + Z_2 \end{vmatrix}} = \frac{Z_1 + Z_2}{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3} V$$

so that

$$V_o = \frac{(Z_1 + Z_2) Z_3}{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3} V$$

and

$$H(s) = \frac{V_o(s)}{V(s)} = \frac{(R_1 + 1/Cs) R_2}{R_1/Cs + R_2/Cs + R_1 R_2}$$

$$\boxed{H(s) = \frac{R_1 R_2 C s + R_2}{R_1 R_2 C s + (R_1 + R_2)}}$$

Note that this problem could have be done using a voltage-divider approach

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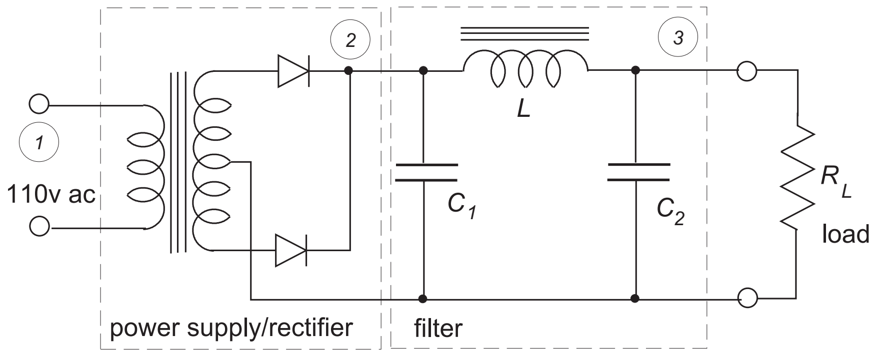
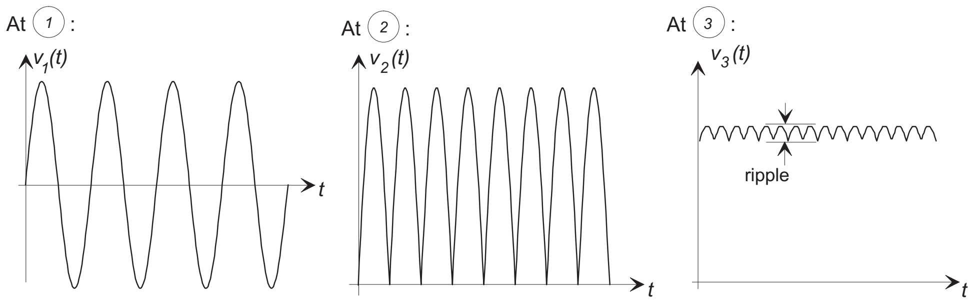
■ Example 5

A common full-wave wave rectified dc power supply for electronic equipment is:

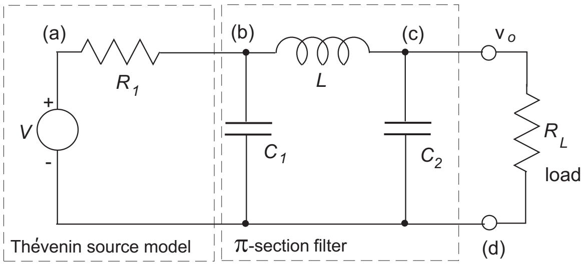


Typical waveforms in the circuit are



The filter acts to "smooth" the full-wave rectified waveform at **(2)** to produce a dc output at **(3)** with a very much reduced "ripple".

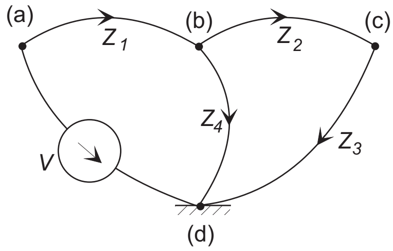
We use a *linearized model* of the transformer/rectifier circuit, with a voltage source (with a waveform as at **(2)** above) and a series resistor R (a Thévenin source) - see Lecture 8 - as below:



The task is to find the transfer function

$$H(s) = \frac{V_o(s)}{V(s)}.$$

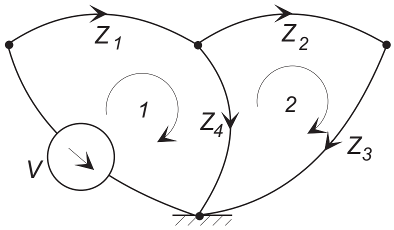
Solution: Combine series and parallel impedances to simplify the structure. Draw as an impedance graph



where

$$\begin{aligned} Z_1 &= R_1 \\ Z_2 &= sL \\ Z_3 &= \frac{1}{sC_2} \parallel R_L = \frac{R_L/sC_2}{R_L + 1/sC_2} \\ Z_4 &= \frac{1}{sC_1} \end{aligned}$$

The system output is $V_{Z_3} = i_{Z_3} Z_3$.
Choose mesh loops to contact all branches as below.



The loop equations are:

$$\begin{aligned} v_{Z_1} + v_{Z_4} - V &= 0 \\ v_{Z_2} + v_{Z_2} - v_{Z_4} &= 0 \end{aligned}$$

and the branch currents (in terms of the mesh currents) are:

$$\begin{aligned} i_{Z_1} &= i_1 \\ i_{Z_2} &= i_2 \\ i_{Z_3} &= i_2 \\ i_{Z_4} &= i_1 - i_2 \end{aligned}$$

Rewrite the loop equations in terms of the mesh currents:

$$\begin{aligned} Z_1 i_1 + Z_4(i_1 - i_2) &= V \\ Z_2 i_2 + Z_3 i_2 - Z_4(i_1 - i_2) &= 0 \end{aligned}$$

or

$$\begin{aligned} (Z_1 + Z_4)i_1 - Z_4 i_2 &= V \\ -Z_4 i_1 + (Z_2 + Z_3 + Z_4)i_2 &= 0 \end{aligned}$$

giving a pair of simultaneous algebraic equations in the mesh currents. Solve for $i_{Z_3} = i_2$

$$\begin{bmatrix} Z_1 + Z_4 & -Z_4 \\ -Z_4 & Z_2 + Z_3 + Z_4 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} V \\ 0 \end{bmatrix}$$

and using Cramer's Rule:

$$i_2 = \frac{\begin{vmatrix} (Z_1 + Z_4) & V \\ -Z_4 & 0 \end{vmatrix}}{\begin{vmatrix} (Z_1 + Z_4) & -Z_4 \\ -Z_4 & Z_2 + Z_3 + Z_4 \end{vmatrix}} = \frac{Z_4 V}{(Z_1 + Z_4)(Z_2 + Z_3 + Z_4) - Z_4^2}$$

and since $V_o = Z_3 i_2$

$$H(s) = \frac{V_o}{V(s)} = \frac{Z_3 Z_4}{Z_1 Z_2 + Z_1 Z_3 + Z_1 Z_4 + Z_2 Z_4 + Z_3 Z_4}$$

Substituting the impedances of the branches

$$\boxed{H(s) = \frac{R_L}{LC_1C_2R_oR_1s^3 + L(C_1R_1 + R_oC_2)s^2 + R_oR_1(C_1 + C_2)s + (R_1 + R_2)}}$$
