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2.004 Dynamics and Control II
Spring 2008

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Massachusetts Institute of Technology
Department of Mechanical Engineering

2.004 *Dynamics and Control II*
Spring Term 2008

Lecture 9[1]

Reading:

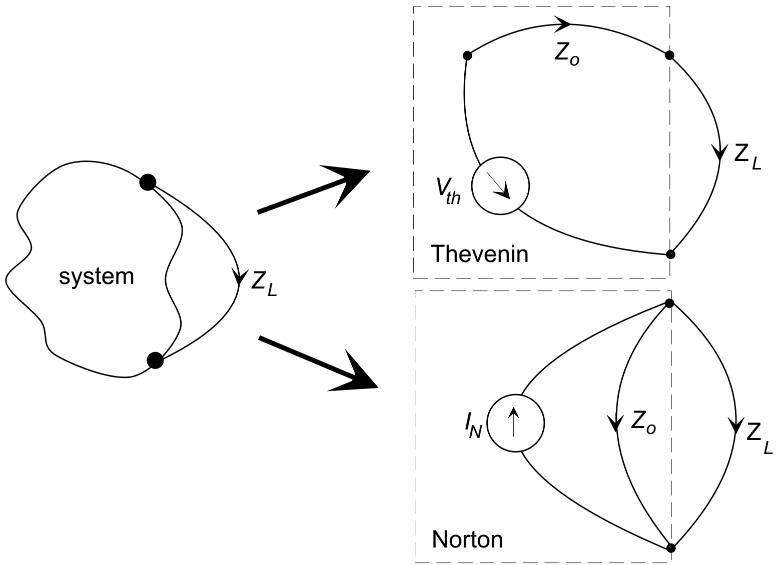
- Nise: Sec. 2.5

1 Thévenin and Norton Source Models

We state the following without proof:

Any linear system (regardless of its internal complexity) containing a single source (voltage or current), and with an external load element Z_L, may be modeled as either
(a) A voltage source V_{th} with a series impedance Z_o (a Thevénin equivalent circuit), or

(b) A current source I_N with a parallel impedance Z_o (a Norton equivalent circuit).



The arbitrary system shown above can be represented in either form, where

Z_o – is the system output impedance, and is the same in each case.

V_{th} – is the Thévenin source voltage – found by removing Z_o and measuring the "open-circuit" terminal voltage.

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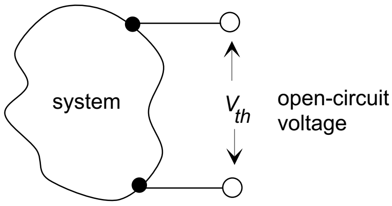
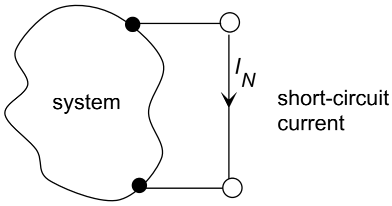
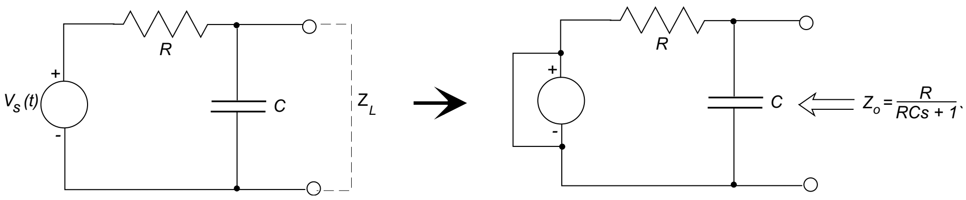


I_N – is the Norton source current – found by "short-circuiting" the output and measuring the current.



To find the output impedance Z_o – set the internal source to zero, and measure the input impedance at the system's output terminals. Some care must be taken in setting any source to zero:

(a) To set a voltage source to zero, short-circuit it, for example to find Z_o in the circuit below:

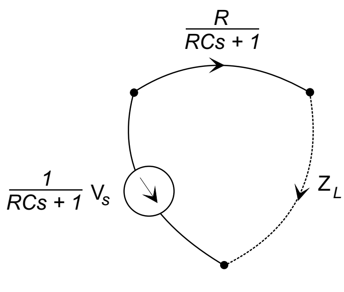


$$Z_o = \frac{1}{Cs} \| R = \frac{R}{CRs + 1}$$

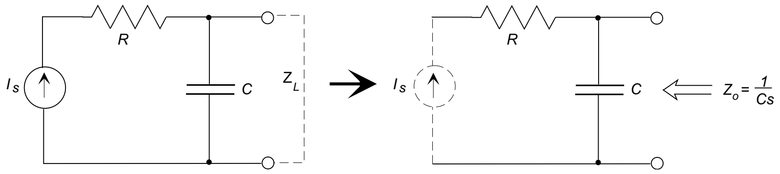
The Thévenin source voltage is found by recognizing the left-hand circuit as a voltage-divider and finding the terminal voltage:

$$V_{th} = \frac{1/(Cs)}{R + 1/(Cs)} V_s = \frac{1}{RCs + 1} V_s$$

The complete Thévenin equivalent for this system is:



(b) To set a current source to zero, remove it from the circuit as shown below:



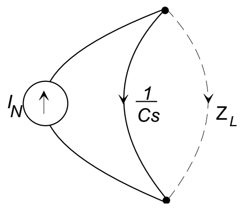
In this case the output impedance seen at the output terminals is

$$Z_o = \frac{1}{Cs}$$

and the Norton short circuit current is simply

$$I_N = I$$

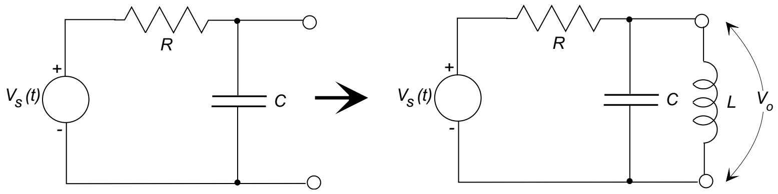
(Note that R does not appear in the system formulation.) The Norton equivalent is:



■ Example 1

Use Thévenin and Norton source models to find the transfer function of the following system when the load is an inductor L.

$$H(s) = \frac{V_o(s)}{V(s)}$$

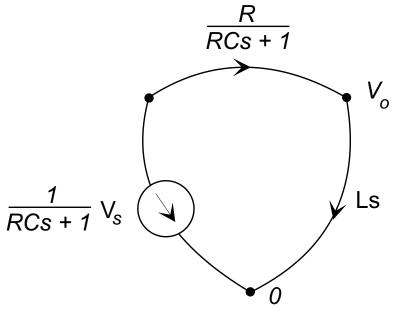


Thévenin Model: For the source model on the left

$$V_{th} = \frac{1/(Cs)}{R + 1/(Cs)} = \frac{1}{RCs + 1}$$

$$Z_o = \frac{1}{Cs} \| R = \frac{R}{CRs + 1}$$

Then for the full system on the right



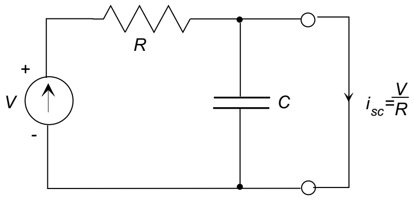
Using a voltage divider relationship

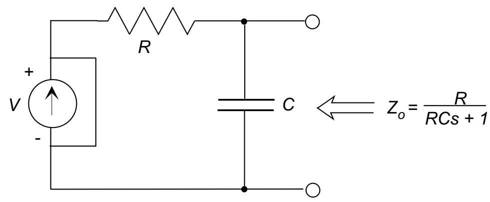
$$\begin{aligned} V_o(s) &= \frac{Ls}{Ls + R/(CRs+1)} V_{th}(s) \\ &= \frac{Ls(RCs+1)}{RLCs^2 + Ls + R} . \frac{1}{RCs+1} V(s). \end{aligned}$$

The transfer function is

$$\boxed{H(s) = \frac{V_o(s)}{V(s)} = \frac{Ls}{RLCs^2 + Ls + R}}$$

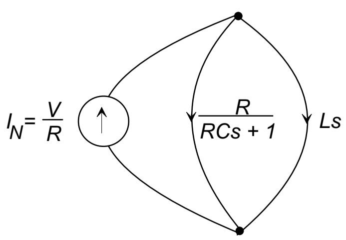
Norton Model: For the Norton model





$$I_N = i_{sc} = \frac{1}{R} V, \qquad Z_o = R \| \frac{1}{Cs} = \frac{R}{RCs+1}$$

and the equivalent system model is



Then the transfer function is found from

$$\begin{aligned} V_o(s) &= \left(Z_o \| \frac{1}{Cs} \right) I_N(s) \\ &= \frac{RLs}{Ls + R/(RCs+1)} \frac{1}{R} V(s) \\ &= \frac{Ls}{RLCs^2 + Ls + R} V \end{aligned}$$

As with the Thévenin method, the transfer function is

$$\boxed{H(s) = \frac{V_o(s)}{V(s)} = \frac{Ls}{RLCs^2 + Ls + R}}$$

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2 Modeling Mechanical Systems

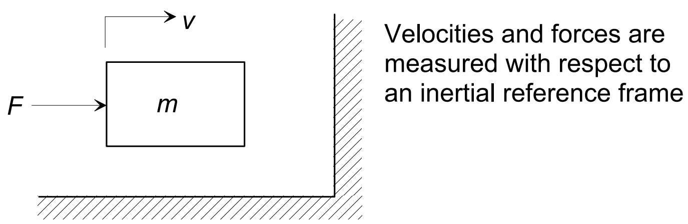
We now move our attention to deriving models of mechanical systems with motion in one dimension. The approach will be to draw analogies with the electrical modeling methods so as to have a unified technique that can be applied without regard to the energy domain.

(1) Choice of Modeling Variables: As in the electrical domain, we select a pair of variables whose product is power, that is we choose F – force (N), and v – velocity (m/s) since

$$P = Fv$$

(2) Modeling Elements: As in the electrical domain we find there are two energy storage elements, and one dissipative element.

(a) The mass element.



For a mass element m

$$F_m = \frac{\mathrm{d}p}{\mathrm{d}t} = m\frac{\mathrm{d}v_m}{\mathrm{d}t}$$

where p is the momentum. We will use the elemental relationship

$$\boxed{\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{1}{m}F} \quad \text{or} \quad \boxed{V(s) = \frac{1}{ms}F(s)}$$

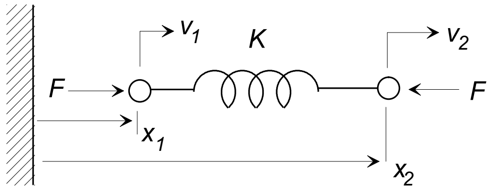
The energy stored in a moving mass element is

$$E = \int_{-\infty}^{t} Pv\,\mathrm{d}t = \frac{1}{2}mv^2$$

which is a function of velocity v and is stored as kinetic energy.

Note: Inertial forces and velocities must be measured with respect to a non-accelerating inertial reference frame. In this course we will assume a reference velocity $v_{ref} = 0$.

(b) The spring (compliance) element:



Let $x_{spring} = (x_2 - x_1) - x_o$ where x_o is the "rest length" of the spring. Then

$$v_{spring} = \frac{\mathrm{d}x_{spring}}{\mathrm{d}t} = \frac{\mathrm{d}x_2}{\mathrm{d}t} - \frac{\mathrm{d}x_1}{\mathrm{d}t} = v_2 - v_1$$

From Hooke's law,

$$F = Kx_{spring}$$

where K is the stiffness (N/m).

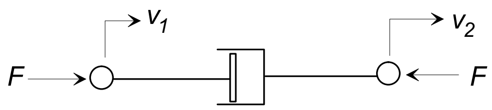
$$\boxed{\frac{\mathrm{d}F_K}{\mathrm{d}t} = Kv_K} \quad \text{or} \quad \boxed{F_K(s) = \frac{K}{s}v_K(s)}$$

The energy stored in a moving mass element is

$$E = \int_{-\infty}^{t} Pv\,\mathrm{d}t = \frac{1}{2K}F_K^2$$

which is a function of velocity F_K and is stored as potential energy.

(c) The viscous friction (dissipative) element: (Also known as the dashpot or damper element)



Let $V_B = v_2 - v_1$, the elemental equation is

$$\boxed{F_B = Bv_B} \quad \text{or} \quad \boxed{F_B(s) = Bv_B(s)}$$

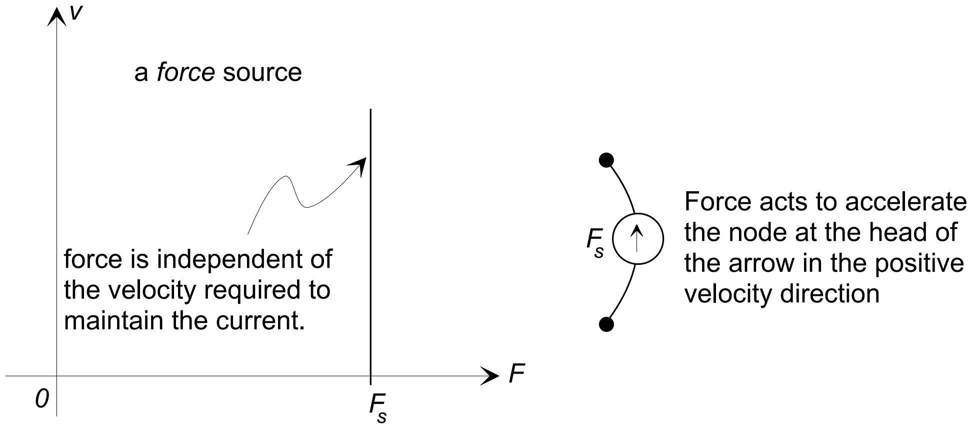
The power flow is

$$P = Fv = Bv^2 \geq 0$$

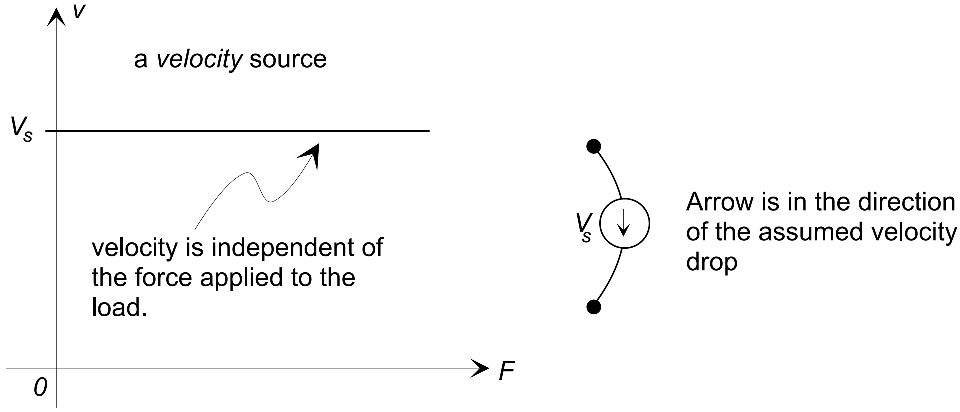
so that the power flow is uni-directional and cannot be recovered.

(3) Ideal Sources: We define a pair of ideal sources

(a) Force Source: The force source maintains a prescribed force F_s regardless of the velocity at which it travels.

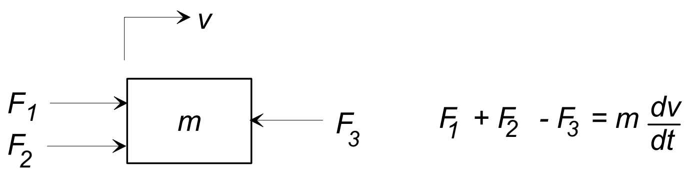


(b) Velocity Source: The velocity source maintains a prescribed velocity V_s regardless of the force required to maintain that velocity.



Interconnection Rules:

(a) Continuity: Consider a mass element with n *external* forces acting on it:



$$\sum_{i=1}^{n} F_i = m\frac{\mathrm{d}v_m}{\mathrm{d}t}$$

or

$$\sum_{i=1}^{n} F_i - m\frac{\mathrm{d}v_m}{\mathrm{d}t} = 0$$

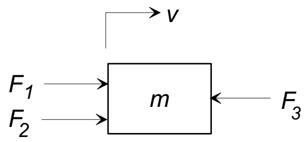
If we substitute a fictitious force $F_m = m\,\mathrm{d}v_m/\,\mathrm{d}t$ (known as a d'Alembert force) we van write

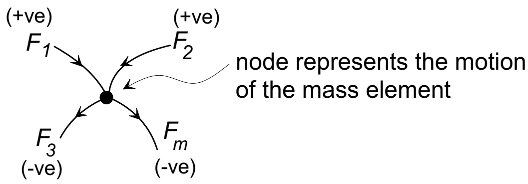
$$\sum_{i=1}^{n} F_i - F_m = 0$$

and state the following

> The sum of all forces acting on a mass element (including the d'Alembert force) is zero.

If we represent a point in space containing a mass element as a node on a graph, and the forces acting on the node as branches, where an arrow pointing at the node means that positive force acts to accelerate the node in the reference direction.





We can write

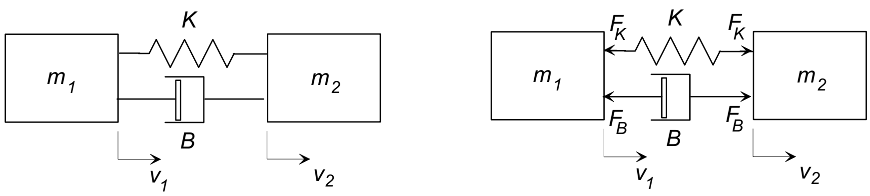
$$F_1 + F_2 - F_3 = F_m = 0$$

The node represents a point in a mechanical system with a distinct velocity.

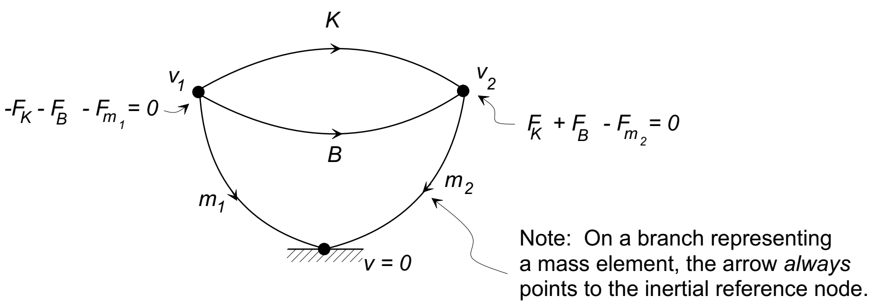
> The continuity condition for a mechanical system is equivalent to Kirchoff's current law in an electrical system.

■ Example 2

The system



may be drawn as a graph:



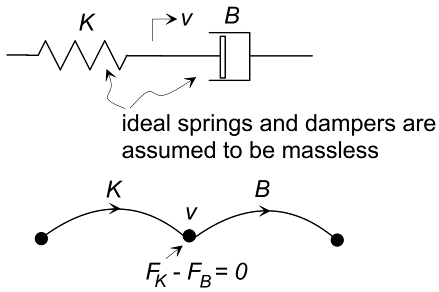
Notice the arrow directions on the branches - as implied by the right hand figure above. The graph implies a pair of continuity equations at the two nodes:

$$\begin{aligned} -F_K - F_B - F_{m_1} &= 0 \qquad \text{or} \qquad -F_K - F_B = m_1 \frac{\mathrm{d}v_1}{\mathrm{d}t} \\ F_K + F_B - F_{m_2} &= 0 \qquad \text{or} \qquad F_K + F_B = m_2 \frac{\mathrm{d}v_2}{\mathrm{d}t} \end{aligned}$$

For a "massless" node (that is at the interconnection of n ideal massless elements)

$$\sum_{i=1}^{n} F_i = 0$$

For example



implies $F_K - F_B = 0$ at the junction.