2.004 Dynamics and Control II
Spring 2008

Massachusetts Institute of Technology
Department of Mechanical Engineering

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Lecture 10[1]

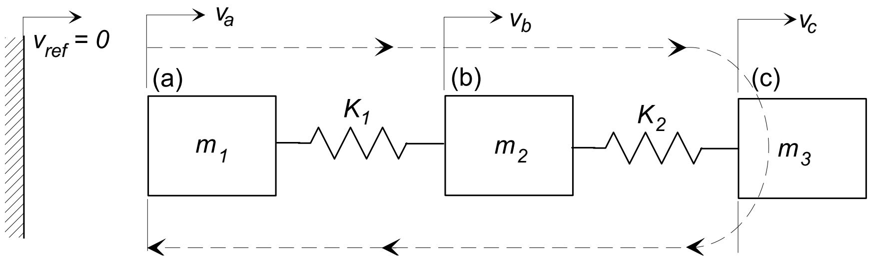
Reading:

- Nise: Sec. 2.5 (pages 59–66)

1 Modeling Mechanical Systems (continued)

In the previous lecture we examined the node (*continuity*) equations for mechanical systems.

Compatibility Condition: Consider the following system:



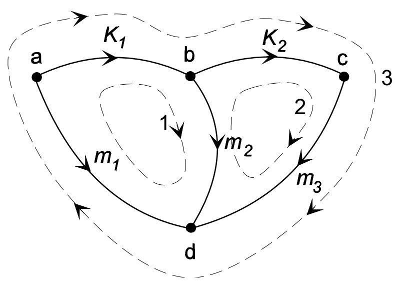
We define the velocity drop between two points in a system as the difference in the velocities (measured with respect to the reference velocity), at two points in the system. For example

$$v_{ab} = v_a - v_b.$$

If we move from node to node around a loop, ending at the starting node, summing the velocity drops as we go, for example a loop from (a) → (b)→ (c)→ (a), and sum the velocity drops

$$v_{ab} + v_{bc} + v_{ca} = (v_a - v_b) + (v_b - v_c) + (v_c - v_a) = 0$$

The graph for the above system is



and around the three loops

$$
\begin{aligned}
v_{K_1} + v_{m_2} - v_{m_3} = (v_a - v_b) + (v_b - v_d) - (v_a - v_d) &= 0 \qquad \text{(Loop 1)}\\
v_{K_2} + v_{m_3} - v_{m_2} = (v_b - v_c) + (v_c - v_d) - (v_b - v_d) &= 0 \qquad \text{(Loop 2)}\\
v_{K_1} + v_{K_2} + v_{m_3} - v_{m_3} = (v_a - v_b) + (v_b - v_c) + (v_c - v_d) - (v_c - v_d) &= 0 \qquad \text{(Loop 3)}
\end{aligned}
$$

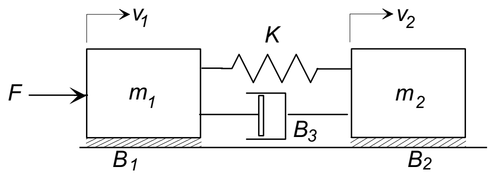
In all cases the sum of the velocity drops is zero. The *compatibility condition* for mechanical systems states:

The sum of velocity drops, from node to node around any closed loop on a system graph is zero.

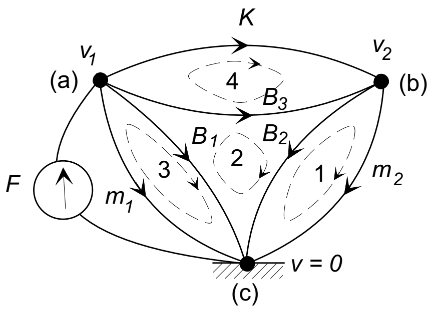
which is analogous to Kirchoff's voltage law for an electrical system.

■ Example 1

Write compatibility equations, and continuity equations for the following system:



The system graph, with four loops defined, is



The compatibility equations for the four loops are:

$$
\begin{aligned}
v_{m_2} - v_{B_2} &= 0 \qquad \text{(Loop 1)}\\
v_{B_2} - v_{B_1} + v_{B_3} &= 0 \qquad \text{(Loop 2)}\\
v_{m_1} - v_{B_2} &= 0 \qquad \text{(Loop 3)}\\
v_K - v_{B_3} &= 0 \qquad \text{(Loop 4)}
\end{aligned}
$$

Continuity equations at nodes (a) and (b) are

$$
\begin{aligned}
F(t) - F_K - F_{B_3} - F_{B_1} - F_{m_1} &= 0 \qquad \text{(Node(a))}\\
F_K + F_{B_3} - F_{B_1} - F_{m_2} &= 0 \qquad \text{(Node(a))}
\end{aligned}
$$

Notes:

1) Branches associated with mass elements **always** connect to the inertial reference node because (i) forces are measured with respect to the inertial reference frame, and (ii) velocities are measured with respect to the inertial reference frame.

2) The arrow on a branch associated with a mass element **always** points away from the node (toward the reference node) because of the sign on the d'Alembert force in the continuity equation

$$\sum_{i=1}^{n} F_i - F_m = 0.$$

Analogy with Electrical Systems: We can compare the interconnection rules defined by system graphs for electrical and mechanical systems:

Electrical:	Currents into a junction (node) sum to zero (KCL).
Mechanical:	Forces at a point (node) (including the d'Alembert force) sum to zero.
Electrical:	Voltage drops around a closed loop sum to zero (KVL).
Mechanical:	Velocity drops around a closed loop in a mechanical system sum to zero.

We use these similarities to make the following analogies between variables in the two energy domains:

$$\boxed{\text{electrical}\left\{\begin{array}{lcl}\text{voltage} & \longleftrightarrow & \text{velocity}\\ \text{current} & \longleftrightarrow & \text{force}\end{array}\right\}\text{mechanical}}$$

Note: The opposite analogies can be made, and are in fact used by many authors, however the above grouping is particularly convenient for use in the graph based method we are developing.

Mechanical Impedance: With the above analogy we define the mechanical impedance as the ratio of velocity to force

$$\boxed{Z_{mech} = \frac{V(s)}{F(s)}} \qquad \left[\text{compare with } Z_{elect} = \frac{V(s)}{Is}.\right]$$

We can also define mechanical admittance

$$Y_{mech} = \frac{1}{Z_{mech}} = \frac{F(s)}{V(s)}$$

Elemental Impedances:

(a) mass element:

$$F_J(s) = msV_J(s) \longrightarrow \quad \boxed{Z_J = \frac{V_J(s)}{F_J(s)} = \frac{1}{ms}}$$

(b) spring element:

$$sF_K(s) = KV_K(s) \longrightarrow \quad \boxed{Z_K = \frac{V_K(s)}{F_K(s)} = \frac{s}{K}}$$

(c) damper element:

$$F_B(s) = BV_B(s) \longrightarrow \quad \boxed{Z_B = \frac{V_B(s)}{F_B(s)} = \frac{1}{B}}$$

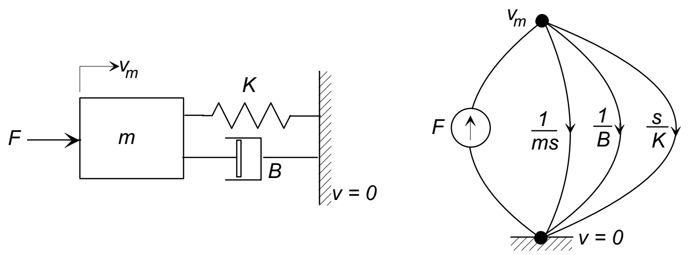
The rules for combining series and parallel mechanical impedances are the same as for electrical impedances, leading to the same methods for generating transfer functions.

■ Example 2

Use impedance methods to find the transfer function

$$H(s) = \frac{V(s)}{F(s)}$$

for



The impedance of the three passive elements is

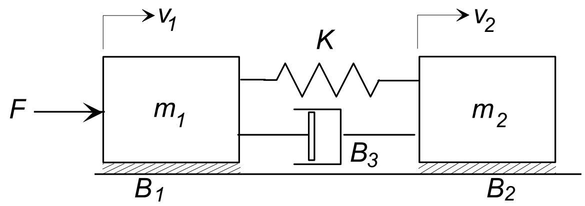
$$\frac{1}{Z_{eq}} = \frac{1}{Z_J} + \frac{1}{Z_K} + \frac{1}{Z_B} = ms + \frac{K}{s} + B = \frac{ms^2 + Bs + K}{s}$$

The transfer function is

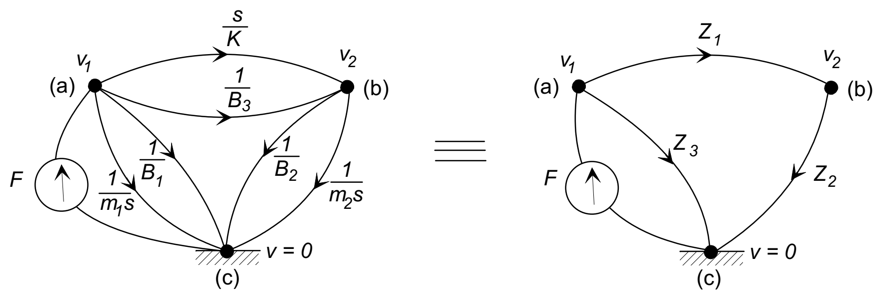
$$H(s) = \frac{V(s)}{F(s)} = Z_{eq} = \frac{s}{ms^2 + Bs + K}$$

■ Example 3

Find the transfer function relating the velocity of the mass m_2 to the input force $F(t)$ in the system:



The system graph on the left may be simplified:



where

$$\begin{aligned} Z_1 = \frac{s}{K} \| \frac{1}{B_3} &= \frac{s}{B_3 s + K} \\ Z_2 = \frac{1}{sm_2} \| \frac{1}{B_2} &= \frac{1}{m_2 s + B_2} \\ Z_3 = \frac{1}{sm_1} \| \frac{1}{B_1} &= \frac{1}{m_1 s + B_1} \end{aligned}$$

We need to compute the velocity at node (a)

$$V_1(s) = F(s) Z_{eq} = F(s) \left(Z_3 \| (Z_1 + Z_2) \right) = \frac{Z_3(Z_1 + Z_2)}{Z_1 + Z_2 + Z_3} F(s)$$

The output velocity $V_2(s)$ can be found using the "velocity divider"

$$V_2(s) = \frac{Z_2}{Z_1 + Z_2} V_1(s) = \frac{Z_2 Z_3}{Z_1 + Z_2 + Z_3} F(s)$$

$$H(s) = \frac{V_2(s)}{F(s)} = \frac{Z_2 Z_3}{Z_1 + Z_2 + Z_3} = \frac{B_3 s + K}{a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

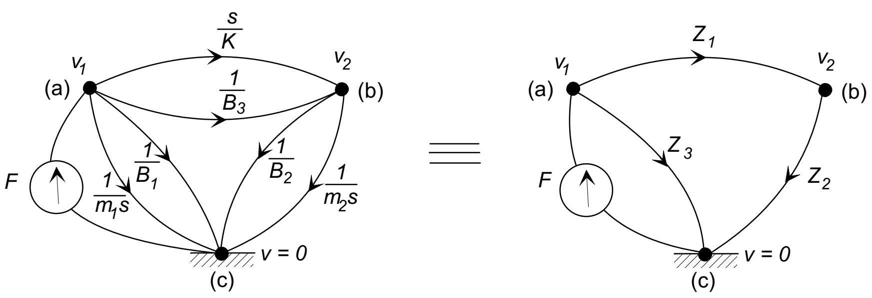
where symbolic software was used to find

$$\begin{aligned} a_3 &= m_1 m_2 \\ a_2 &= m_1(B_2 + B_3) + m_2(B_1 + B_3) \\ a_1 &= K(m_1 + m_2) + B_1 B_2 + B_1 B_3 + B_2 B_3 \\ a_0 &= K(B_1 + B_2) \end{aligned}$$

■ Example 4

Repeat Example 4 using *node equations* to find $H(s)$.

Solution: From Example 4, the impedance graph, and a simplified form are



where

$$\begin{aligned} Z_1 = \frac{s}{K} \| \frac{1}{B_3} &= \frac{s}{B_3 s + K} \\ Z_2 = \frac{1}{sm_2} \| \frac{1}{B_2} &= \frac{1}{m_2 s + B_2} \\ Z_3 = \frac{1}{sm_1} \| \frac{1}{B_1} &= \frac{1}{m_1 s + B_1} \end{aligned}$$

Write a pair of node equations expressing the continuity conditions at (a) and (b):

$$\begin{aligned} F - F_{Z_1} - F_{Z_2} &= 0 \qquad \text{at node (a)} \\ F_{Z_2} - F_{Z_3} &= 0 \qquad \text{at node (b)} \end{aligned}$$

For convenience, use admittances instead of impedances. Let $Y_1 = 1/Z_1$, $Y_2 = 1/Z_2$, and $Y_3 = 1/Z_3$. Substitute for the admittance relationships ($F = Yv$) on each branch:

$$\begin{aligned} Y_1(v_a - 0) + Y_2(v_a - v_b) &= F \qquad \text{at node (a)} \\ Y_2(v_a - v_b) - Y_3(V_b - 0) &= 0 \qquad \text{at node (b),} \end{aligned}$$

which are a pair of simultaneous linear equations in v_a and v_b:

$$\begin{bmatrix} Y_1 + Y_2 & -Y_2 \\ Y_2 & -(Y_2 + Y_3) \end{bmatrix} \begin{bmatrix} v_a \\ v_b \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}$$

and may be solved using Cramer's rule:

$$v_b(s) = v_{m_1}(s) = \frac{\begin{vmatrix} Y_1+Y_2 & F \\ Y_2 & 0 \end{vmatrix}}{\begin{vmatrix} Y_1+Y_2 & -Y_2 \\ Y_2 & -(Y_2+Y_3) \end{vmatrix}} = \frac{Y_2 F(s)}{Y_1Y_2 + Y_2Y_3 + Y_3Y_1}$$

$$= \frac{Z_1 Z_3}{Z_1 + Z_2 + Z_3} F(s)$$

by dividing numerator and denominator by $Y_1Y_2Y_3$. Then

$$\boxed{H(s) = \frac{x_{m_1}(s)}{F(s)} = \frac{1}{s}\frac{v_{m_1}(s)}{F(s)} = \frac{1}{s}\frac{Z_1Z_3}{Z_1+Z_2+Z_3},}$$

which is the same result found using ad-hoc impedance reduction methods in Example 4.