2.004 Dynamics and Control II
Spring 2008

Massachusetts Institute of Technology
Department of Mechanical Engineering

2.004 *Dynamics and Control II*
Spring Term 2008

Lecture 14[1]

Reading:

- Class Handout: *Modeling Part 1: Energy and Power Flow in Linear Systems* Sec. 3.
- Class Handout: *Modeling Part 2: Summary of One-Port Primitive Elements*

1 The Modeling of Rotational Systems.

With the the modeling framework as we defined it in Lecture 13, we have seen that in each energy domain we need to define

(a) Two power variables, an *across variable*, and a *through variable*. the product of these variables is power.

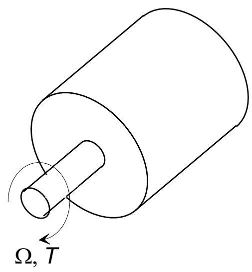
(b) Two ideal sources, and *across variable source*, and a *through variable source*.

(c) Three ideal modeling elements, two energy storage elements (a T-type element, and a A-Type element), and a dissipative (D-Type) element.)

(d) A pair of interconnection laws.

We now address modeling of rotational mechanical systems.

(a) Definition of Power variables: In a rotational system we consider the motion of a system around an *axis of rotation*:



Consider the rotary motion resulting from a force F applied at a radius r from the rotational axis

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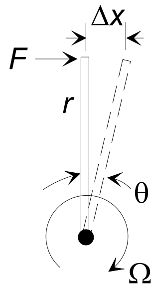


The work done by the force F in moving an infinitesimal distance Δx is

$$\Delta W = F\Delta x = Fr\theta$$

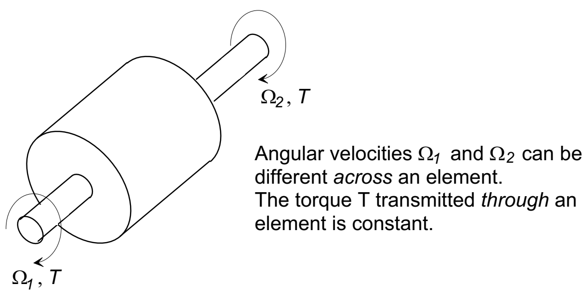
and the power P is

$$\boxed{P = \frac{\mathrm{d}\Delta W}{\mathrm{d}t} = Fr\frac{\mathrm{d}\theta}{\mathrm{d}t} = T\Omega}$$

where $T = Fr$ is the applied torque (N.m), and $\Omega = \mathrm{d}\theta/\mathrm{d}t$ is the angular velocity (rad/s).

We note that if T and Ω have the same sign, then $P > 0$ and power is flowing into the system or element that is being rotated. Similarly, if T and Ω have the opposite signs, then $P < 0$ and power is flowing from the system or element, in other words the system is doing work on the source.

Note that the angular velocity Ω can be different across an element, but that torque T is transmitted through an element:

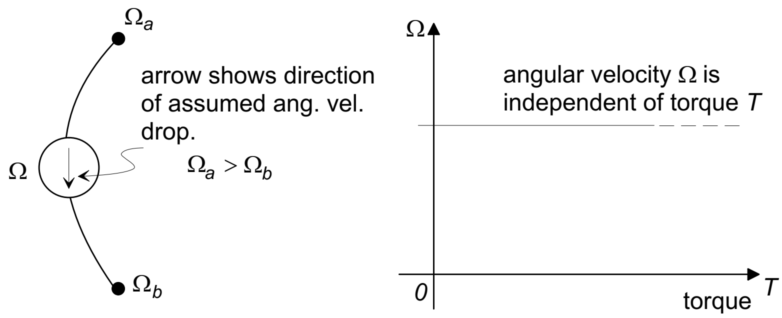


We therefore define our power variables as torque T and angular velocity Ω, where

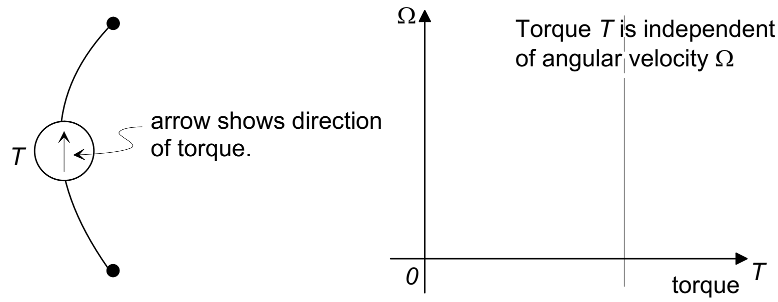
- T is chosen as the *through* variable
- Ω is chosen as the *across* variable.

(b) Ideal Sources: With the choice of modeling variables we can define our pair of ideal sources

The Angular Velocity Source: $\Omega_s(t)$
By definition the angular velocity source is an *across variable source.*The ideal angular velocity source will maintain the rotational speed regardless of the torque it must generate to do so:

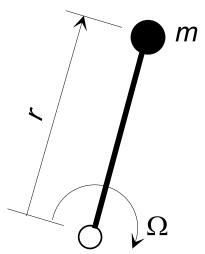


The Torque Source: $T_s(t)$
By definition the torque source is a *through variable source.* The ideal torque source will maintain the applied torque regardless of the angular velocity it must generate to do so:



(c) Ideal Modeling Elements:

1 The Moment of Inertia: Consider a mass element m rotating at a fixed radius R about the axis of rotation.



The stored energy is

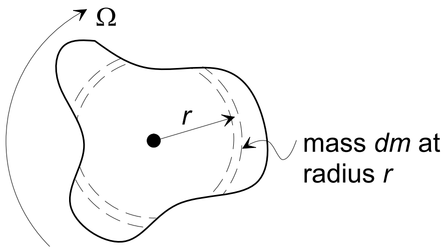
$$E = \frac{1}{2}m(r\Omega)^2 = \frac{1}{2}J\Omega^2$$

where $J = mr^2$ is defined to be the moment of inertia of the particle.

For a collection of n mass particles m_i at radii r_i, $i = 1, \ldots, n$, the moment of inertia is

$$J = \sum_{i=1}^{n} m_i r_i^2.$$

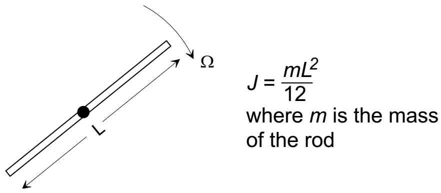
For a continuous distribution of mass about the axis of rotation, the moment of inertia is



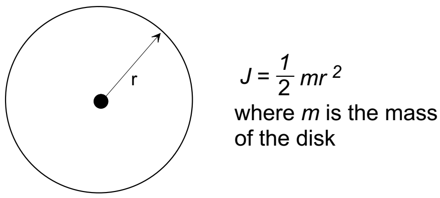
$$J = \int_0^R r^2 dm$$

Examples:

A uniform rod of length L rotating about its center.



A uniform disc with radius r rotating about its center.



The elemental equation for the moment of inertia J is

$$\boxed{T_J = J\frac{\mathrm{d}\Omega_J}{\mathrm{d}t}}$$

We note that the energy stored in a rotating mass is $E = J\Omega^2/2$, that is it is a function of the across variable, defining the moment of inertia as an *A-type element.*

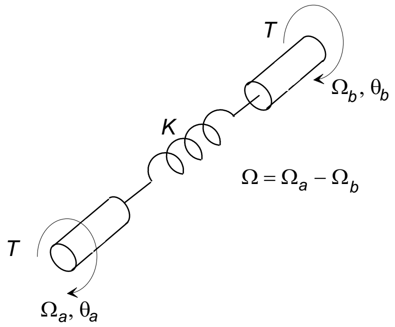
As in the case of a translational mass element, the angular velocity drop associated with a rotary inertia J is *always measured with respect to a non-accelerating reference frame.*

Elemental Impedance: By definition

$$\boxed{Z_J = \frac{\Omega_J(s)}{T_J(s)} = \frac{1}{Js}}$$

from the elemental equation.

(2) The Torsional Spring:



Let θ_a and θ_b be the angular displacements of the two ends from their rest positions. Hooke's law for a torsional spring is

$$T = K(\theta_a - \theta_b).$$

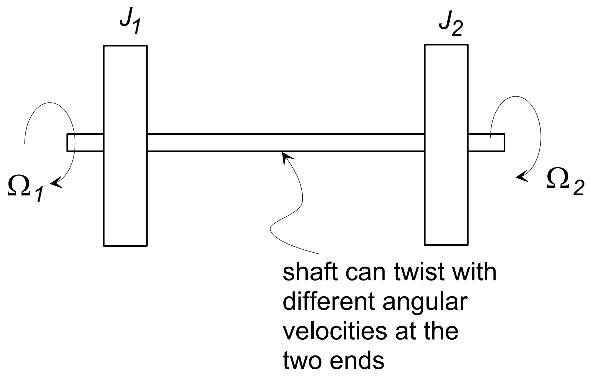
where K is defined to be the *torsional stiffness*. Differentiation gives

$$\frac{\mathrm{d}T}{\mathrm{d}t} = K\frac{\mathrm{d}(\theta_a - \theta_b)}{\mathrm{d}t}$$

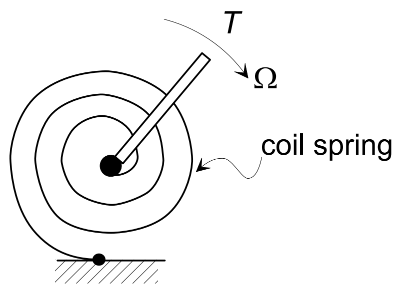
$$\boxed{\frac{\mathrm{d}T}{\mathrm{d}t} = K\Omega}$$

where $\Omega = (\dot{\theta}_a - \dot{\theta}_b)$ is the *angular velocity drop* across the spring.

Torsional stiffness may result from the material properties of a "long" shaft



or may be intentional, for example in a coil ("hair") spring in a mechanical watch.



The energy stored in a torsional spring is

$$E = \int_{\infty}^{t} T\Omega \, \mathrm{d}t = \frac{1}{2K}T^2$$

which is a function of the through variable, defining the spring as a *T-type element*.

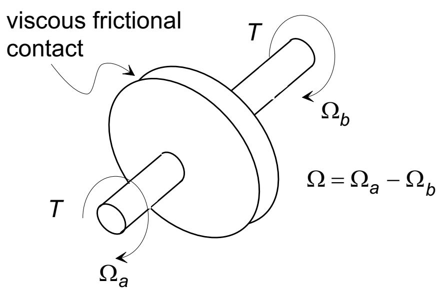
Elemental Impedance: By definition

$$\boxed{Z_K = \frac{\Omega_K(s)}{T_K(s)} = \frac{s}{K}}$$

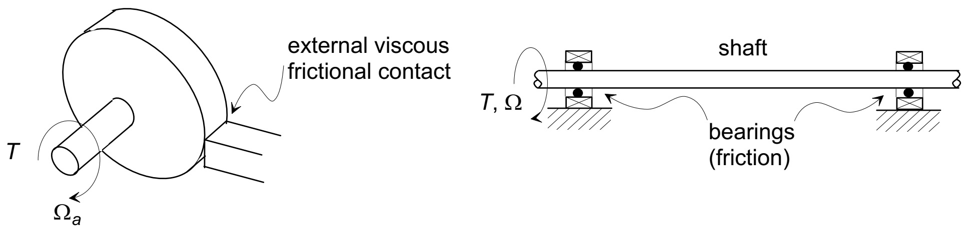
from the elemental equation.

(3) The Rotational Damper: We look for an algebraic relationship between T and Ω of the form

$$\boxed{T = B\Omega}$$



which is approximated as viscous rotational friction:



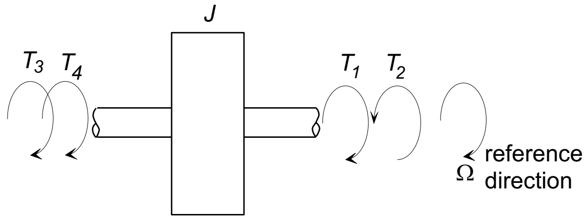
Notice that $P = T\Omega = B\Omega^2 > 0$, which defines the damper as a D-type element.

Elemental Impedance: By definition

$$\boxed{Z_B = \frac{\Omega_B(s)}{T_B(s)} = \frac{1}{B}}$$

from the elemental equation.

(d) Interconnection Laws: Consider an inertial element J subject to n external torques $T_1, T_2, \ldots, T_n$, for example



then

$$J\frac{\mathrm{d}\Omega}{\mathrm{d}t} = T_1 - T2 + T_3 + T_4$$

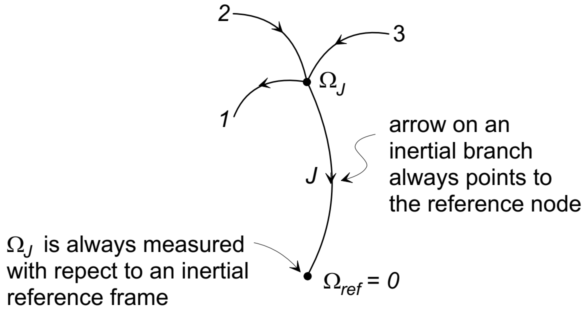
and in general

$$\sum_{i=1}^{n} T_i = J\frac{\mathrm{d}\Omega}{\mathrm{d}t}$$

As in the translational case, we consider a "fictitious" d'Alembert torque T_j and write

$$\boxed{\sum_{i=1}^{n} T_i - T_J = 0}$$

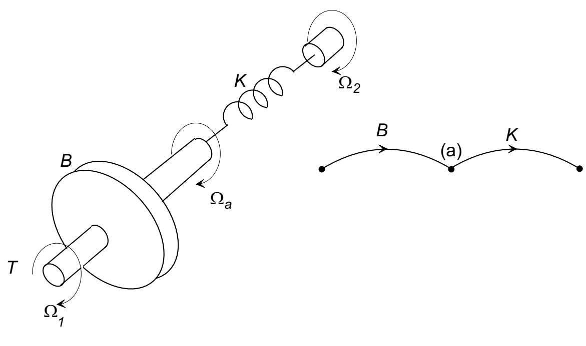
as the torque balance (*continuity condition*) at a node.



For an "inertia-less" node ($J = 0$),

$$\sum_{i=1}^{n} T_i = 0$$

which states that the external torques sum to zero, for example at node (a) below, $T_B - T_K = 0$.

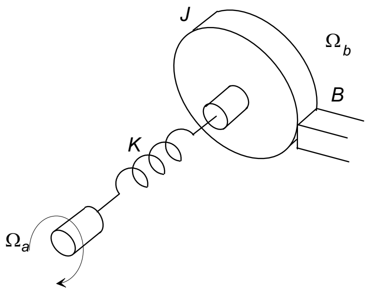


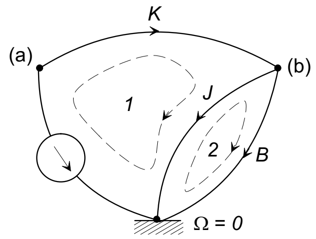
Continuity Condition: The sum of torques (including a d'Alembert torque associated with an inertia a element) at any node on a system graph is zero.

Nodes represent points of distinct angular velocity in a rotational system, and by analogy with translational systems, the compatibility condition is

Compatibility Condition: The sum of angular velocity drops around any closed loop on a system graph is zero.

For example, on the graph:





two compatibility equations are:

$$\begin{aligned}\Omega_K + \Omega_J - \Omega_s &= 0 \qquad \text{(Loop 1)},\\ \Omega_B - \Omega_J &= 0 \qquad \text{(Loop 2)}.\end{aligned}$$

2 Updated Tables of Generalized Elements to Include Rotational Elements:

The tables presented in Lecture 13 are now updated to include rotational systems.

A-Type Elements:

Element	Elemental equation	Energy
Generalized A-type	$\mathsf{f} = \mathsf{C}\frac{d\mathsf{v}}{dt}$	$\mathcal{E} = \frac{1}{2}\mathsf{C}\mathsf{v}^2$
Translational mass	$F = m\frac{dv}{dt}$	$\mathcal{E} = \frac{1}{2}mv^2$
Rotational inertia	$T = J\frac{d\Omega}{dt}$	$\mathcal{E} = \frac{1}{2}J\Omega^2$
Electrical capacitance	$i = C\frac{dv}{dt}$	$\mathcal{E} = \frac{1}{2}Cv^2$

T-Type Elements :

Element	Elemental equation	Energy
Generalized T-type	$\mathsf{v} = \mathsf{L} d\mathsf{f}/dt$	$\mathcal{E} = \frac{1}{2}\mathsf{L}\mathsf{f}^2$
Translational spring	$v = \frac{1}{K}\frac{dF}{dt}$	$\mathcal{E} = \frac{1}{2K}F^2$
Torsional spring	$\Omega = \frac{1}{K}\frac{dT}{dt}$	$\mathcal{E} = \frac{1}{2K}T^2$
Electrical inductance	$v = L\frac{di}{dt}$	$\mathcal{E} = \frac{1}{2}Li^2$

D-Type Elements:

Element	Elemental equations		Power dissipated
Generalized D-type	$\mathsf{f} = \frac{1}{\mathsf{R}}\mathsf{v}$	$\mathsf{v} = \mathsf{R}\mathsf{f}$	$\mathcal{P} = \frac{1}{\mathsf{R}}\mathsf{v}^2 = \mathsf{R}\mathsf{f}^2$
Translational damper	$F = Bv$	$v = \frac{1}{B}F$	$\mathcal{P} = Bv^2 = \frac{1}{B}F^2$
Rotational damper	$T = B\Omega$	$\omega = \frac{1}{B}T$	$\mathcal{P} = B\Omega^2 = \frac{1}{B}T^2$
Electrical resistance	$i = \frac{1}{R}v$	$v = Ri$	$\mathcal{P} = \frac{1}{R}v^2 = Ri^2$

Generalized Impedances:

	A-Type	T-Type	D-Type
Generalized	$\frac{1}{Cs}$	sL	R
Translational	$\frac{1}{sm}$	$\frac{1}{K}s$	$\frac{1}{B}$
Rotational	$\frac{1}{sJ}$	$\frac{1}{K}s$	$\frac{1}{B}$
Electrical	$\frac{1}{Cs}$	sL	R