2.004 Dynamics and Control II
Spring 2008

Massachusetts Institute of Technology
Department of Mechanical Engineering

2.004 *Dynamics and Control II*
Spring Term 2008

Lecture 16[1]

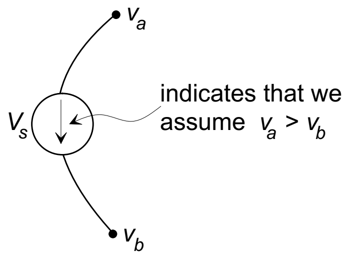
Reading:

- Class Handout - *Modeling Part 3: Two-Port Energy Transducing Elements*

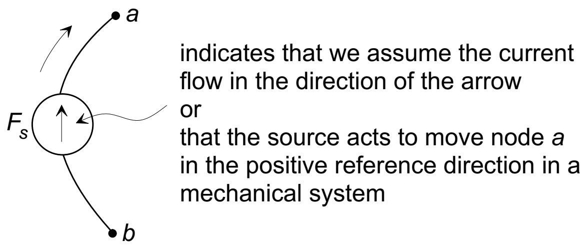
1 Arrow Conventions on Ideal Sources

To this point we have simply told you to draw the arrows on source elements

(a) In the direction of the assumed across-variable drop for across-variable sources (voltage and velocities),



(b) in the direction of the assumed through-variable direction for through-variable sources (currents and forces)



When we draw a branch on a graph we make the assumption that $P > 0$, that is that power is flowing into the element. There are in fact two arrows implicit on each branch: one representing the assumed across-variable drop, and a second representing the assumed through-variable direction. If $P > 0$ (power is flowing into the element), the two arrows are in the same direction. For example, consider a capacitor

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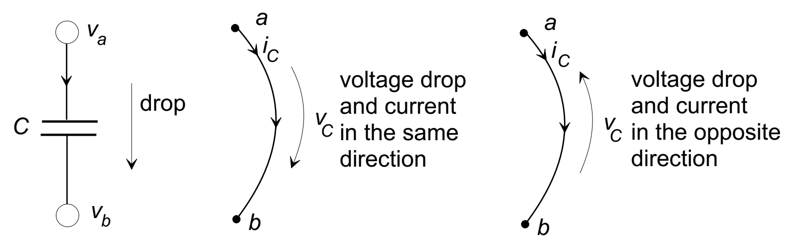


$P = v_c i_c > 0$ when either

1. $v_c > 0$ and $i_c > 0$,or
2. $v_c < 0$ and $i_c < 0$.

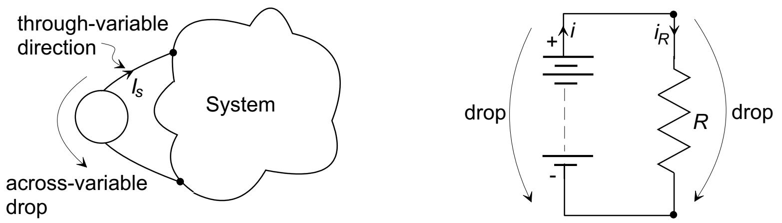
$P = v_c i_c < 0$ when either

1. $v_c > 0$ and $i_c < 0$,or
2. $v_c < 0$ and $i_c > 0$.

On passive elements, with the assumption $P > 0$ we can combine the two arrows into one because they point in the same direction.

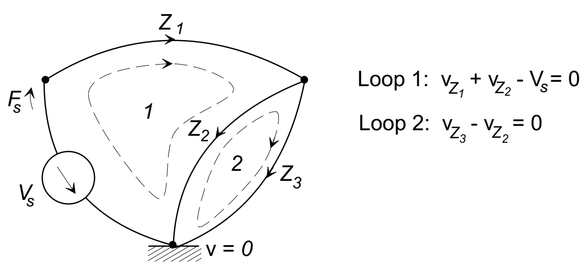
For sources, the assumption is the opposite - that is we assume that the source is supplying energy/power to the system, $P < 0$.

There are two arrows (pointing in opposite directions) associated with any source – one for the across-variable, the second for the through-variable.

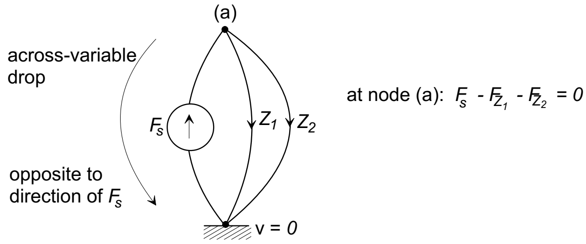


In modeling sources, we choose to show the arrow that will normally be used to solve the system;

(a) An across-variable source will usually be included in a loop-equation, therefore the convention is to show the arrow associated with the across-variable drop.



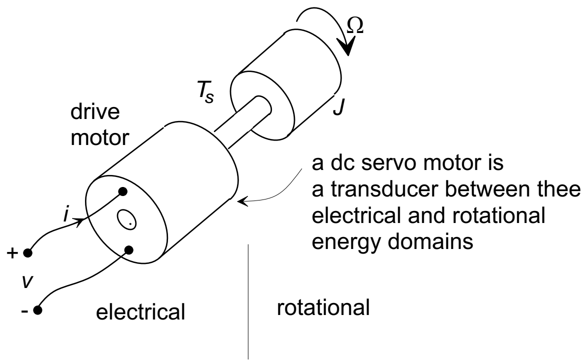
(b) A through-variable source will be included in a node-equation, therefore the convention is to show the arrow representing the direction of the assumed through variable.



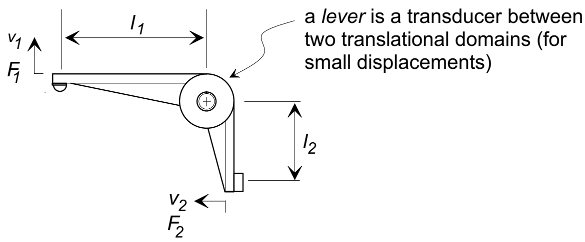
2 Energy Transduction – Two-Port Elements

Reading: Class Handout - *Modeling Part 3: Two-Port Energy Transducing Elements*

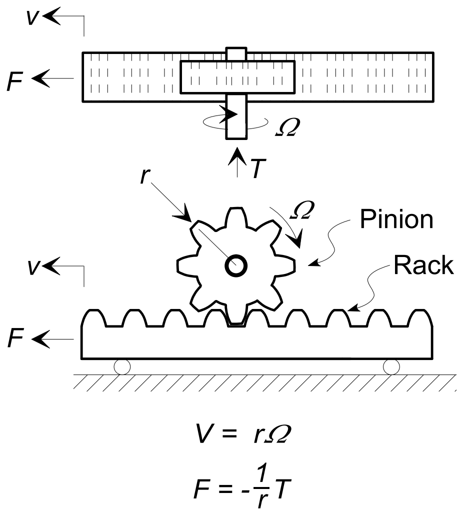
Many systems involve two or more energy domains, for example a system containing a dc motor



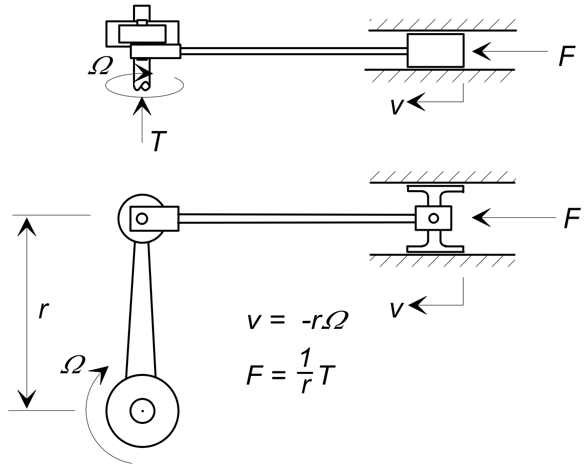
or there may be a scaling of the across- and through-variables within a single domain, for example a mechanical lever



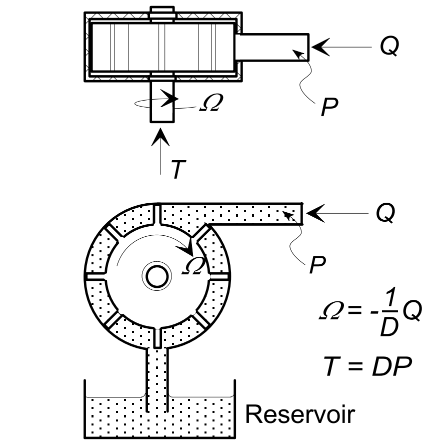
The following two pages show some examples of two-port elements.



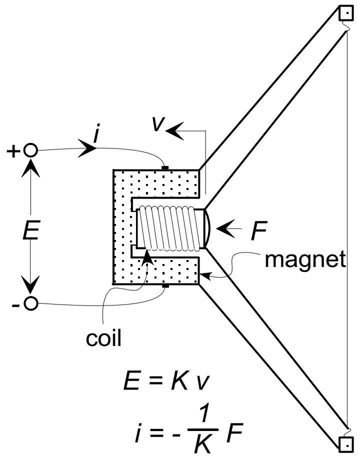
(a) Rack and pinion



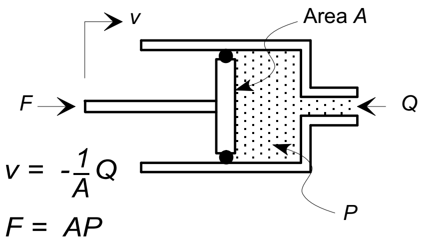
(b) Slider-crank



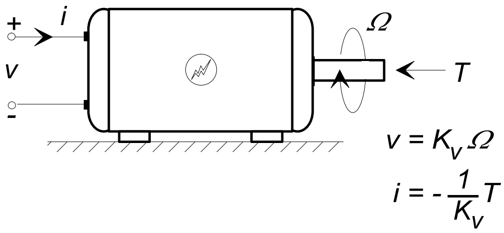
(c) Rotary positive displacement pump



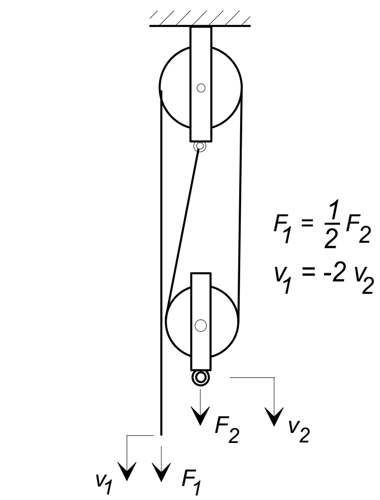
(d) Moving-coil loudspeaker



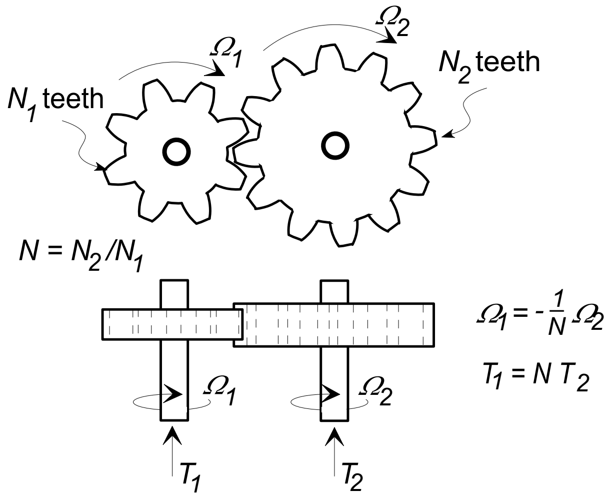
(d) Fluid piston-cylinder



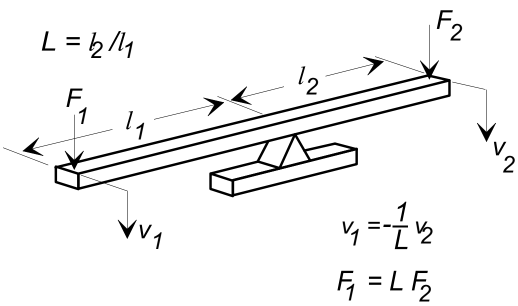
(b) Electrical motor/generator



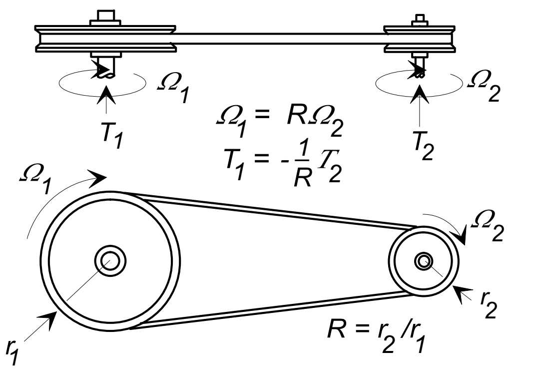
(a) Block and tackle



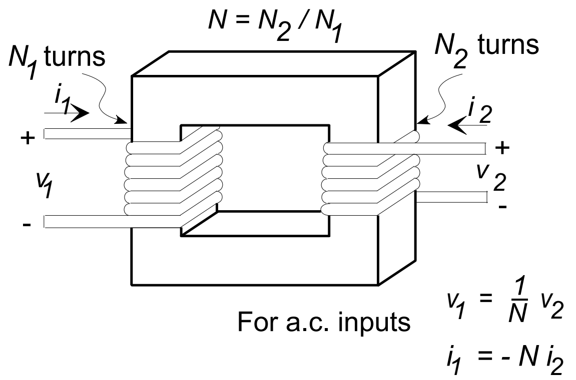
(b) Gear train



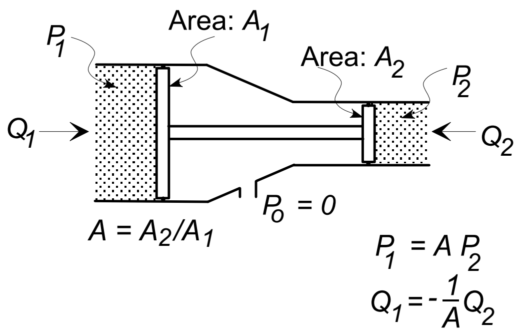
(c) Mechanical lever



(d) Belt drive

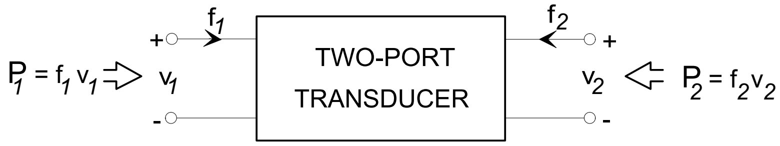


(e) Electrical transformer



(f) Fluid transformer

In all of these examples the energy transduction is lossless and static, that is there is no energy storage.



$$P_1 + P_2 = f_1 v_1 + f_2 v_2 = 0$$

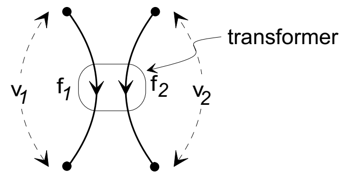
where the power flow at each port is defined to be positive *into* the port.

There are two possibilities:

(a) There is an proportional relationship between the across-variables on the two sides of the two-port element, and an proportional relationship between the through-variables.

$$\begin{aligned} v_1 &= k v_2 && \text{across-variable}_1 \propto \text{across-variable}_2 \\ f_1 &= (1/k) f_2 && \text{through-variable}_1 \propto \text{through-variable}_2, \end{aligned}$$

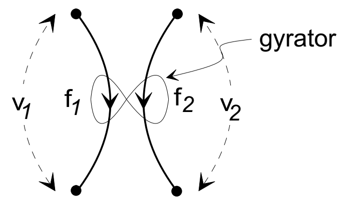
where clearly $P_1 = P_2$. This relationship defines a *transformer*.



(b) There is an proportional relationship between the across-variables on one side and the through-variable on the other side of the two-port element.

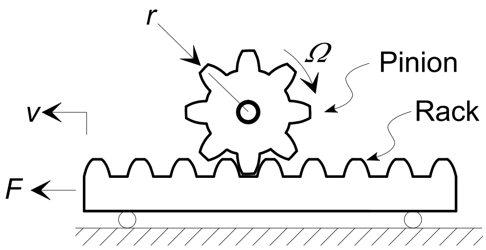
$$\begin{aligned} v_1 &= k f_2 && \text{across-variable}_1 \propto \text{through-variable}_2 \\ f_1 &= (1/k) v_2 && \text{through-variable}_1 \propto \text{across-variable}_2, \end{aligned}$$

where clearly $P_1 = P_2$. This relationship defines a *gyrator*.



Examples:

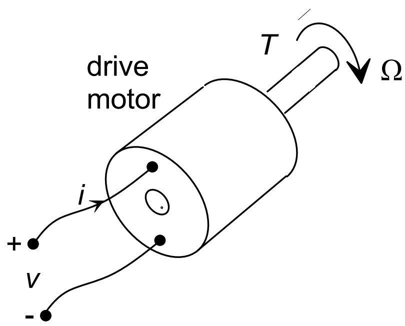
(1) Rack and Pinion:



$$V = r\Omega, \quad F = -\frac{1}{r}T \quad \Big\}\ \text{a transformer}$$

It can be seen that the linear velocity of the rack is proportional to the angular velocity of the pinion. Similarly the force needed to balance the torque applied to the pinion is proportional to the torque. The rack and pinion is therefore a lossless *transformer.*

(2) DC motor:

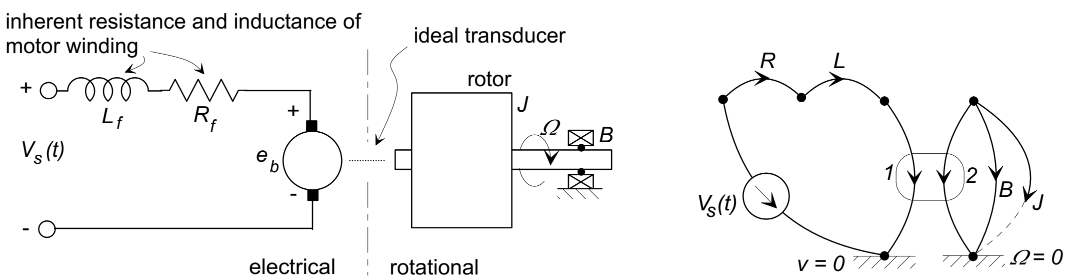


$$T = -Ki, \quad \Omega = \frac{1}{K}v \quad \Big\}\ \text{a transformer}$$

In the dc motor the torque produced is proportional to the current flowing, while the back emf produced is proportional to the angular velocity of the shaft. The motor is a lossless *transformer.*

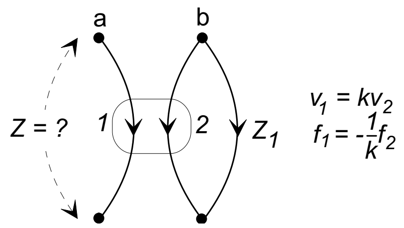
■ Example 1

A DC motor with an inertial load



Impedance Relationships across Two-Port Elements

(1) The Transformer:



Let an element (or system) with impedance Z_1 be connected across a transformer as shown, then the impedance seen from side 1 of the transformer is

$$Z = \frac{V_1(s)}{F_1(s)}.$$

At node (b)

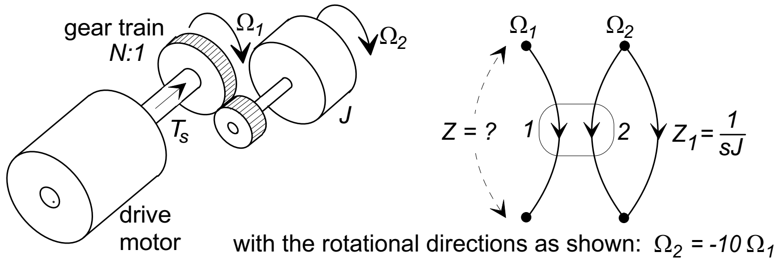
$$\begin{aligned} f_2 &= -f_{Z_1} \\ v_2 &= v_{Z_1} \end{aligned}$$

so that

$$\boxed{Z = \frac{V_1(s)}{F_1(s)} = \frac{kV_2(s)}{(-1/k)F_2(s)} = k^2\frac{V_{Z_1}(s)}{F_{Z_1}(s)} = k^2 Z_1}$$

■ Example 2

Find the apparent inertia of an inertia J with a step-gear box with a 10:1 ratio.

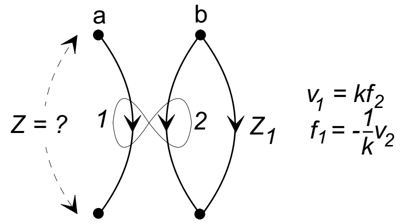


With the above definition $k = \Omega_1/\Omega_2 = -0.1$, so that

$$Z_{in} = k^2 Z_1(s) = \frac{0.01}{Js} = \frac{1}{(100J)s}$$

or in other words the inertia J is seen at the input shaft as an equivalent inertia $100J$.

(2) The Gyrator:



At node (b)

$$\begin{aligned} f_2 &= -f_{Z_1} \\ v_2 &= v_{Z_1} \end{aligned}$$

so that

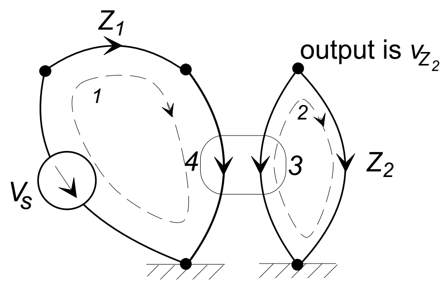
$$\boxed{Z = \frac{V_1(s)}{F_1(s)} = \frac{kF_2(s)}{(-1/k)V_2(s)} = k^2\frac{F_{Z_1}(s)}{v_{Z_1}(s)} = k^2\frac{1}{Z_1} = k^2 Y_1.}$$

The nature of the apparent impedance as seen at the input has been changed to its reciprocal. The result is that an A-type element appears to be a T-type element when connected behind a gyrator, and vice-versa.

No example is given here because there are no naturally occurring gyrators in the energy domains covered this term.

Impedance Based Modeling with Two-Port Elements

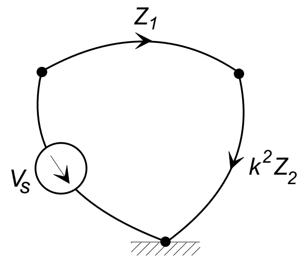
Consider a Thévenin source coupled to a load Z_2 through a transformer.



The transformer provides the constraints

$$\begin{aligned} v_4 &= kv_3 \\ f_4 &= -(1/k)f_3 \end{aligned}$$

(a) If Z_2 is reflected to the l.h. side of the transformer:



$$V_4(s) = \frac{k^2 Z_2}{Z_1 + k^2 Z_2} V_s(s)$$

but

$$V_{Z_2} = v_3 = \frac{1}{k} v_4$$

so that the transfer function $H(s)$ is

$$\boxed{H(s) = \frac{V_{Z_2}}{V_s(s)} = \frac{k Z_2}{Z_1 + k^2 Z_2}.}$$

(b) Alternatively, we can transfer all elements to the r.h. side of the transformer

$$v_{Z_1} = v_3 = \frac{1}{k} v_4$$

From loop (1)

$$v_{Z_1} + v_4 - V_s = 0$$

and

$$\begin{aligned} v_{Z_2} &= \frac{1}{k}\left(v_3 - v_{Z_1}\right) \qquad \text{but } v_{Z_1} = Z_1 f_{Z_1} \\ &= \frac{1}{k}\left(V_s - Z_1 f_4\right) \qquad (f_{Z_1} = f_4) \\ &= \frac{1}{k}\left(V_s - \frac{Z_1}{k Z_2} V_{Z_1}\right) \end{aligned}$$

so that

$$V_{Z_2}\left(1 + \frac{1}{k}\frac{Z_1}{Z_2}\right) = \frac{1}{k} V_s$$

or

$$\boxed{H(s) = \frac{V_{Z_2}}{V_s(s)} = \frac{k Z_2}{Z_1 + k^2 Z_2}.}$$

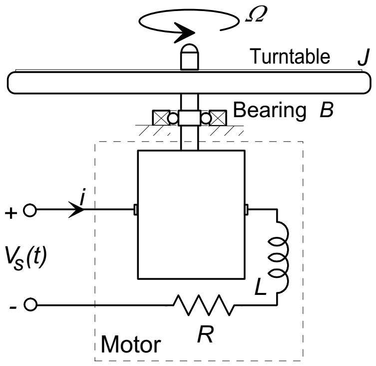
which is the same result as above.

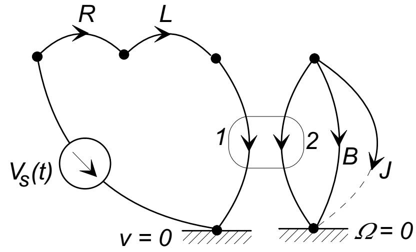
■ Example 3

Use this result to find the transfer function

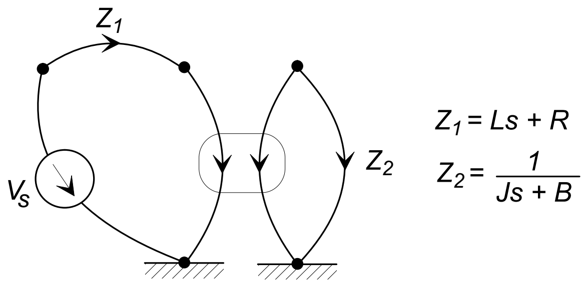
$$H(s) = \frac{\Omega_J(s)}{V_{in}(s)}$$

for the system shown below, which includes a dc servo motor, driven from a voltage source $V_{in}(s)$, driving an inertial load J and bearing damping B





Combine the series and parallel impedances and redraw the graph



where $Z_1 = R + sL$, and $Z_2 = 1/(Js + B)$. Assume that for the motor $v_b = v_1 = K_m\Omega_m$. From the previous result

$$H(s) = \frac{\Omega_J(s)}{V_{in}(s)} = \frac{kZ_2}{Z_1 + k^2 Z_2} = \frac{K_m/(Js+B)}{(R+sL) + K_m^2/(Js+B)}$$

$$\boxed{H(s) = \frac{K_m}{JLs^2 + (RJ + BL)s + (BR + K_m^2)}}$$
