

MIT OpenCourseWare
<http://ocw.mit.edu>

2.004 Dynamics and Control II
Spring 2008

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.

Lecture 17¹

Reading:

- Nise: Chapter 4

1 System Response

Our systems are

- *Linear and Time-Invariant* (LTI), that is they are linear, and their properties do not change with time, and
- are *Single-Input Single-Output* (SISO).

and are usually represented by a transfer function

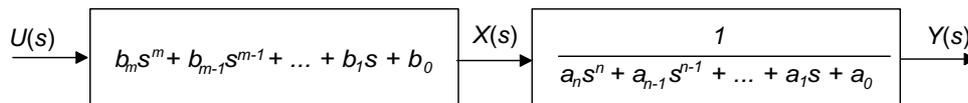
$$U(s) \rightarrow \boxed{\frac{b_n s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}} \rightarrow Y(s)$$

which is equivalent to an *ordinary differential equation* (ODE) with constant coefficients.

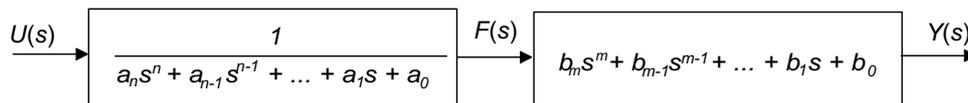
$$a_n \frac{dy^n}{dt^n} + a_{n-1} \frac{dy^{n-1}}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{du^m}{dt^m} + b_{m-1} \frac{du^{m-1}}{dt^{m-1}} + \dots + b_1 \frac{du}{dt} + b_0 u$$

where the coefficients are determined by the system components.

Block diagram algebra allows us to redraw the block diagram as

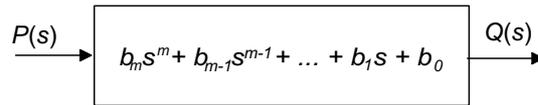


or alternatively



Consider first the action of the block containing the numerator polynomial

¹copyright © D.Rowell 2008

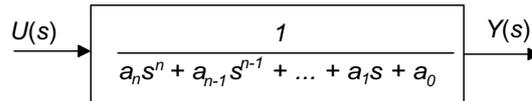


then

$$q(t) = b_m \frac{dp^m}{dt^m} + b_{m-1} \frac{dp^{m-1}}{dt^{m-1}} + \dots + b_1 \frac{dp}{dt} + b_0 p,$$

which *is not* a differential equation. It simply shows that the output of this block is a *weighted sum of the input and its derivatives*.

Initially therefore, we concentrate on solving the sub-system represented by the block



or as a differential equation

$$a_n \frac{dy^n}{dt^n} + a_{n-1} \frac{dy^{n-1}}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y = u.$$

From 18.03 we know that the solution to an ODE of this form may be considered to have two components:

$$y(t) = y_h(t) + y_p(t)$$

where

- $y_h(t)$ is the *solution to the homogeneous equation*, that is the solution when $u(t) \equiv 0$, and
- $y_p(t)$ is a *particular solution* that satisfies the ODE for the given $u(t)$.

a) The Homogeneous Solution $y_h(t)$: Assume that $u(t) \equiv 0$ and that the system has initial conditions $y(0) = C_0$, $\dot{y}(0) = C_1$, $\ddot{y}(0) = C_2$, \dots , $y^{(n)}(0) = C_n$, then conjecture a solution of the form

$$y_h(t) = K e^{\lambda t}$$

so that

$$y^{(m)}(t) = \frac{d^m y_h}{dt^m} = K \lambda^m e^{\lambda t}.$$

Substituting into the homogeneous differential equation gives

$$K (a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0) e^{\lambda t} = 0$$

Because $K \neq 0$, and $e^{\lambda t} \neq 0$ for finite t , we therefore require

$$\boxed{a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0 = 0,}$$

which is the *characteristic equation* of the system.

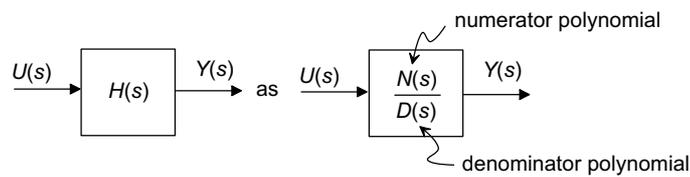
The assumed form of the solution

$$y_h(t) = K e^{\lambda t}$$

is a solution for any value of λ satisfying the characteristic equation. Since the order of the characteristic polynomial is n , there will be n such roots, and the most general form of $y_h(t)$ will be

$$y_h(t) = \sum_{i=1}^n K_i e^{\lambda_i t}$$

where the constants K_i are determined from the initial conditions, and the λ_i are known as the *eigenvalues* (characteristic values) of the system. We note that if we write the transfer function



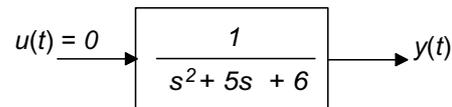
$$H(s) = \frac{N(s)}{D(s)}$$

then the characteristic equation (in terms of s is

$$D(s) = 0.$$

■ Example 1

Find the homogeneous response of the system



when $y(0) = 2$, and $\dot{y}(0) = 1$.

The characteristic equation is

$$\begin{aligned} s^2 + 5s + 6 &= 0 \\ (s + 3)(s + 2) &= 0 \end{aligned}$$

so that

$$s_1, s_2 = -3, -2$$

and the response is

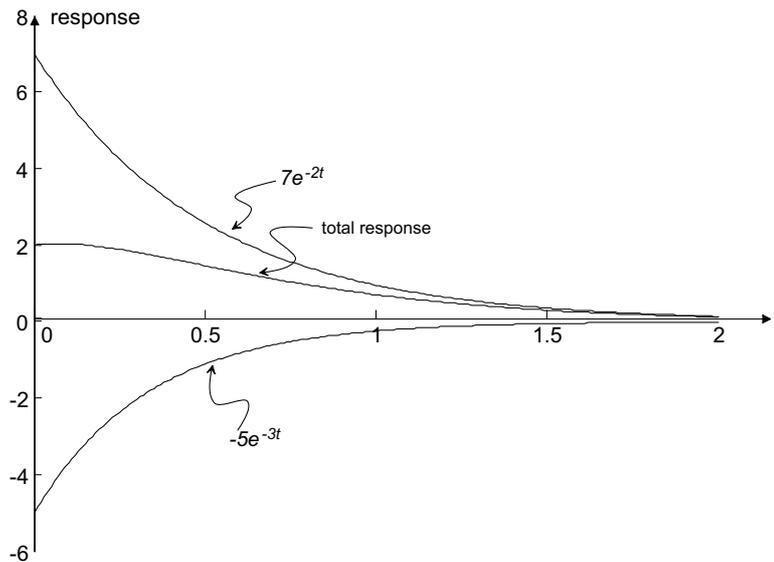
$$y_h(t) = K_1 e^{-3t} + K_2 e^{-2t}.$$

K_1 and K_2 are found from the initial conditions. When $t = 0$:

$$\begin{aligned} y(0) &= 2 = K_1 + K_2 \\ \dot{y}(0) &= 1 = -3K_1 - 2K_2 \end{aligned}$$

from which $K_1 = -5$, and $K_2 = 7$. The complete solution is

$$y_h(t) = -5e^{-3t} + 7e^{-2t}.$$



Notice that this system is stable, that is all exponential components in the homogeneous response decay to zero as time t increases.

a) The Particular Solution $y_p(t)$: The particular solution is often found using the *method of undetermined coefficients*, in which a solution of a given form, with unknown coefficients, is assumed for the given input function $u(t)$, and the coefficients are found to satisfy the differential equation.

The following table summarizes the assumed solution forms for some common input functions:

Term in $u(t)$	Assumed form for $y_p(t)$	Test value
k	K_1	0
$kt^n \quad (n = 1, 2, 3 \dots)$	$K_n t^n + K_{n-1} t^{n-1} + \dots + K_1 t + K_0$	0
$ke^{\lambda t}$	$K_1 e^{\lambda t}$	λ
$ke^{j\omega t}$	$K_1 e^{j\omega t}$	$j\omega$
$k \cos(\omega t)$	$K_1 \cos(\omega t) + K_2 \sin(\omega t)$	$j\omega$
$k \sin(\omega t)$	$K_1 \cos(\omega t) + K_2 \sin(\omega t)$	$j\omega$

■ Example 2

Find the particular solution, and the complete solution for the system

$$H(s) = \frac{1}{s^2 + 5s + 6}$$

when the input $u(t) = 4$ (for $t > 0$) and $y(0) = 0$ and $\dot{y}(0) = 0$.

The differential equation is

$$\frac{d^2 y}{dt^2} + 5 \frac{dy}{dt} + 6y = u(t).$$

From the above table, for a constant input we assume $y_p(t) = K$ and substituting into the differential equation

$$\begin{aligned} \frac{d^2 y_p}{dt^2} + 5 \frac{dy_p}{dt} + 6y_p &= K \\ 0 + 5 \times 0 + 6K &= 4 \end{aligned}$$

or $y_p(t) = K = 2/3$.

From the previous example, $y_h(t) = K_1 e^{-3t} + K_2 e^{-2t}$, so the complete solution is

$$y(t) = y_h(t) + y_p(t) = K_1 e^{-3t} + K_2 e^{-2t} + \frac{2}{3}.$$

With the stated initial conditions ($y(0) = 0$ and $\dot{y}(0) = 0$),

$$\begin{aligned} y(0) &= 0 = K_1 + K_2 + 2/3, \\ \dot{y}(0) &= 0 = -3K_1 - 2K_2 \end{aligned}$$

from which $K_1 = 4/3$, and $K_2 = -2$. The complete solution is therefore

$$y(t) = -\frac{4}{3} e^{-3t} - 2 e^{-2t} + \frac{2}{3}.$$