2.004 Dynamics and Control II
Spring 2008

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING

2.004 *Dynamics and Control II*
Spring Term 2008

Lecture 19[1]

Reading:

- Nise: Chapter 4.

1 System Poles and Zeros

Consider a system with transfer function

$$H(s) = \frac{N(s)}{D(s)}.$$

If we factor the numerator and denominator polynomials and write

$$H(s) = K\frac{(s-z_1)(s-z_2)\dots(s-z_m)}{(s-p_1)(s-p_2)\dots(s-p_n)}$$

where

$p_1, p_2, \dots, p_n$ — are the roots of the characteristic polynomial $D(s)$, and are known as the *system poles*,

$z_1, z_2, \dots, x_m$ — are the roots of the numerator polynomial $N(s)$, and are known as the *system zeros.*

Note that because the coefficients of $N(s)$ and $D(s)$ are *real* (they come from the modeling parameters), the system poles and zeros must be either

(a) purely *real*, or

(b) appear as *complex conjugates*

and in general we write

$$p_i, \text{ or } z_i = \sigma_i + j\omega_i.$$

■ Example 1

Find the poles and zeros of the system

$$\begin{aligned} G(s) &= \frac{5s^2+10s}{s^3+5s^2+11s+5} \\ &= \frac{5s(s+2)}{(s+3)(s^2+2s+5)} \\ &= \frac{5s(s+2)}{(s+3)(s+(1+j2))(s+(1-j2))} \end{aligned}$$

so that we have

(a) a pair of real zeros at $s = 0, -2$ and

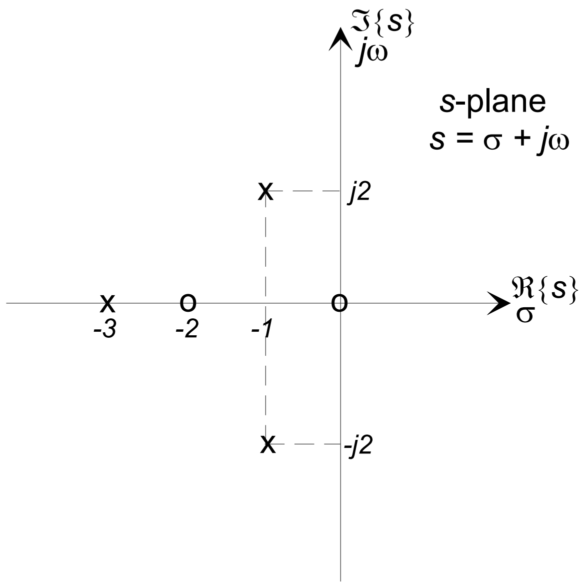
(b) three poles at $s = -3, -1 + j2$, and $s = -1 - j2$.

The system poles and zeros completely characterize the transfer function (and therefore the system itself) except for an overall gain constant K:

$$G(s) = K \frac{\prod_{i=1}^{m} (s - z_i)}{\prod_{i=1}^{n} (s - p_i)}$$

1.1 The Pole-Zero Plot

The values of a system's poles and zeros are often shown graphically on the complex s-plane in a pole-zero plot. For example, the poles and zeros of the previous example are drawn:



where the pole positions are denoted by an **x**, and the zeros are drawn as an **o**. The figure shows zeros at $s = 0, -2$, and poles at $s = -3, -1 \pm j2$.

The pole-zero plot is used extensively throughout control theory and system dynamics to provide a qualitative indication of the dynamic behavior of systems.

Aside: In MATLAB a system may be specified by its poles and zeros using the function `zpk(zeros, poles, gain)`, for example

```
sys = zpk([0 2], [-3, -1+i*2, -1-i*2], 5)
step(sys)
```

will plot the step response of the system in the previous example.

The *characteristic equation* of a system is

$$D(s) = (s - p_1)(s - p_2) \dots (s - p_n) = 0$$

so that the poles are the system eigenvalues, and the form of the homogeneous response is dictated by the poles:

$$y_h(t) = \sum_{i=1}^{n} C_i e^{p_i t}$$

(when the poles are distinct), and the constants C_i are determined by the initial conditions.

■ Example 2

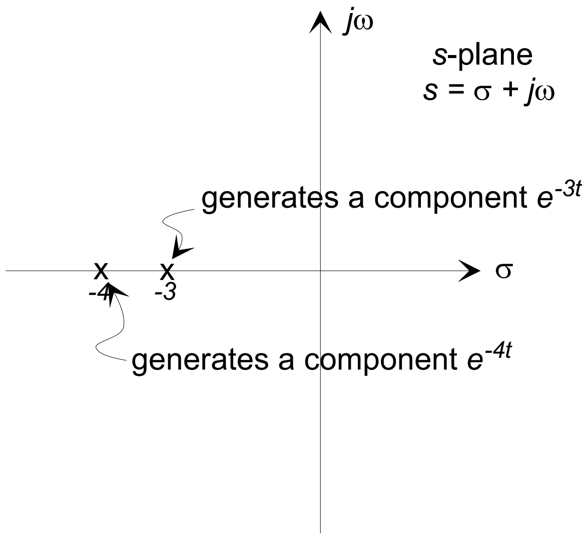
Find the poles and hence the homogeneous response components of the system

$$G(s) = \frac{12}{s^2 + 7s + 12}$$

The characteristic equation is

$$D(s) = (s + 4)(s + 4) = 0$$

and the poles are $s = -3, -4$



The homogeneous response components are therefore $y_1(t) = C_1 e^{-3t}$ and $y_2(t) = C_2 e^{-4t}$, where C_1 and C_2 are defined by the initial conditions.

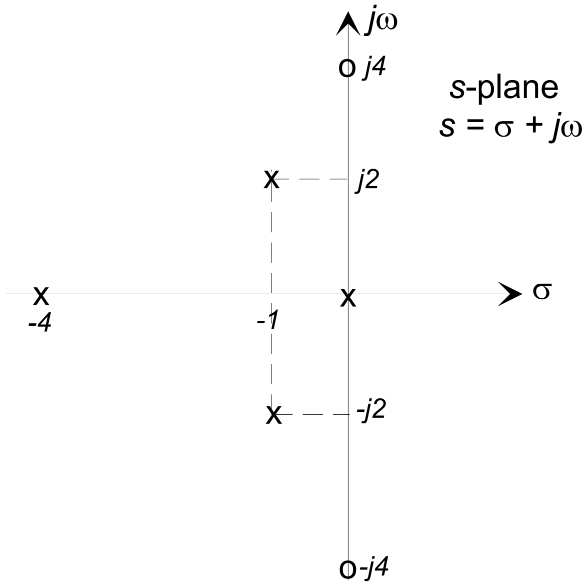
Note: The poles do not specify the amplitude of the modal components in the response. They simply indicate the nature of the response components.

1.2 Complex Poles and Zeros

We have noted that in general $s = \sigma + j\omega$, and that poles and zeros may appear as complex conjugate pairs:

$$\begin{aligned} p_{i,i+1} &= \sigma_i \pm j\omega_i \\ z_{k,k+1} &= \sigma_k \pm j\Omega_k \end{aligned}$$

For example the pole-zero plot



corresponds to the transfer function

$$\begin{aligned} G(s) &= K \frac{(s - j4)(s + j4)}{s(s+4)(s + (1 + j2))s + (1 - j2))} \\ &= K \frac{s^2 + 16}{s(s+4)(s^2 + 2s + 5)} \\ &= K \frac{s^2 + 16}{s^4 + 16s^3 + 13s^2 + 2s} \end{aligned}$$

The homogeneous response we will have a pair of complex exponential terms associated with each pair of conjugate pair of poles, such as

$$\ldots + C_i e^{(\sigma_i + j\omega_i)t} + C_{i+1} e^{(\sigma_i - j\omega_i)t} \ldots$$

but C_i and C_{i+1} are also complex (say $a \pm jb$), so we can write

$$\begin{aligned} C_i e^{(\sigma_i + j\omega_i)t} + C_{i+1} e^{(\sigma_i - j\omega_i)t} &= (a + jb) e^{\sigma_i t} e^{j\omega_i t} + (a - jb) e^{\sigma_i t} e^{-j\omega_i t} \\ &= a e^{\sigma_i t} \left(e^{j\omega_i t} + e^{j\omega_i t} \right) - jb e^{\sigma_i t} \left(e^{j\omega_i t} - e^{j\omega_i t} \right) \end{aligned}$$

Euler's formulas state

$$\left. \begin{aligned} \cos(\omega t) = \tfrac{1}{2} \left(e^{j\omega t} + e^{-j\omega t} \right) \\ \sin(\omega t) = \tfrac{1}{2j} \left(e^{j\omega t} - e^{-j\omega t} \right) \end{aligned} \right\} \quad \text{or} \quad \left\{ \begin{aligned} e^{j\omega t} &= \cos(\omega t) + j \sin(\omega t) \\ e^{-j\omega t} &= \cos(\omega t) - j \sin(\omega t) \end{aligned} \right.$$

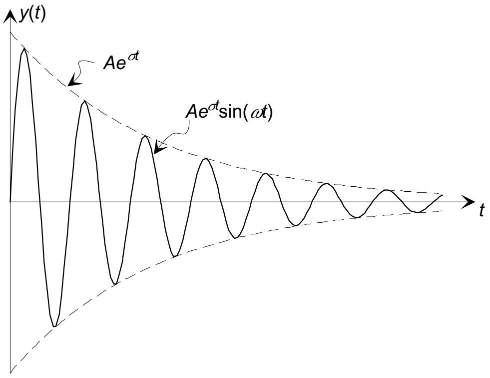
so that the contribution of the complex conjugate pole pair to the homogeneous response may be written

$$\begin{aligned} y_{i,i+1}(t) &= 2ae^{\sigma_i t}\cos(\omega_i t) + 2ae^{\sigma_i t}\sin(\omega_i t)) \\ &= 2\sqrt{a^2+b^2}e^{\sigma_i t}\left(\frac{a}{\sqrt{a^2+b^2}}\cos(\omega_i t) + \frac{b}{\sqrt{a^2+b^2}}\sin(\omega_i t)\right) \\ &= A_i e^{\sigma_i t}\sin(\omega_i t + \phi_i) \end{aligned}$$

where

$$A_i = 2\sqrt{a^2+b^2} \quad \text{and} \quad \phi_i = \tan^{-1}\left(\frac{a}{b}\right).$$

which is shown below (for $\sigma_i < 0$).



■ Example 3

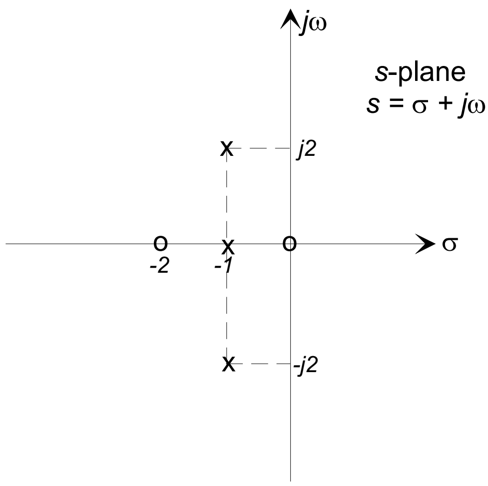
Find and plot the poles and zeros of

$$G(s) = \frac{7s+14}{s^3+3s^2+7s+5}$$

and then determine the modal response components of this system.

$$G(s) = 7\frac{s+2}{(s+1)(s^2+2s+5} = 7\frac{s+2}{(s+1)(s+(1+j2))(s+(1-j2))}$$

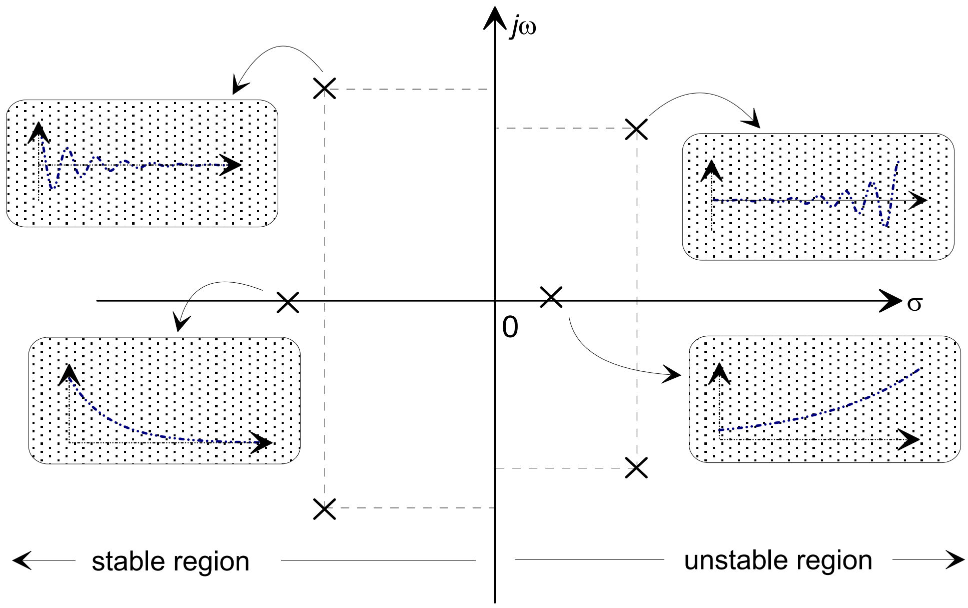
The pole-zero plot is



and the modal components are (1) Ce^{-t} (corresponding to the pole at $s = -1$), and (2) $Ae^{-t}\sin(2t + \phi)$ (corresponding to the complex conjugate pole pair at $s = -1 \pm j2$), and where the constants C, A, and ϕ are determined from the initial conditions.

Note: A pair of purely imaginary poles (on the imaginary axis of the s-plane) implies $\sigma = 0$ and there will be no decay. The system will act as a pure oscillator.

The effect of pole locations in the s-plane on the modal response components is summarized in the figure below:



We note

(a) Poles in the left-half of the s-plane (the lhp), that is $\sigma < 0$, generate components that decay with time.

(b) Conversely, poles in the right-half s-plane (the rhp), that is $\sigma > 0$ generate components that grow with time.

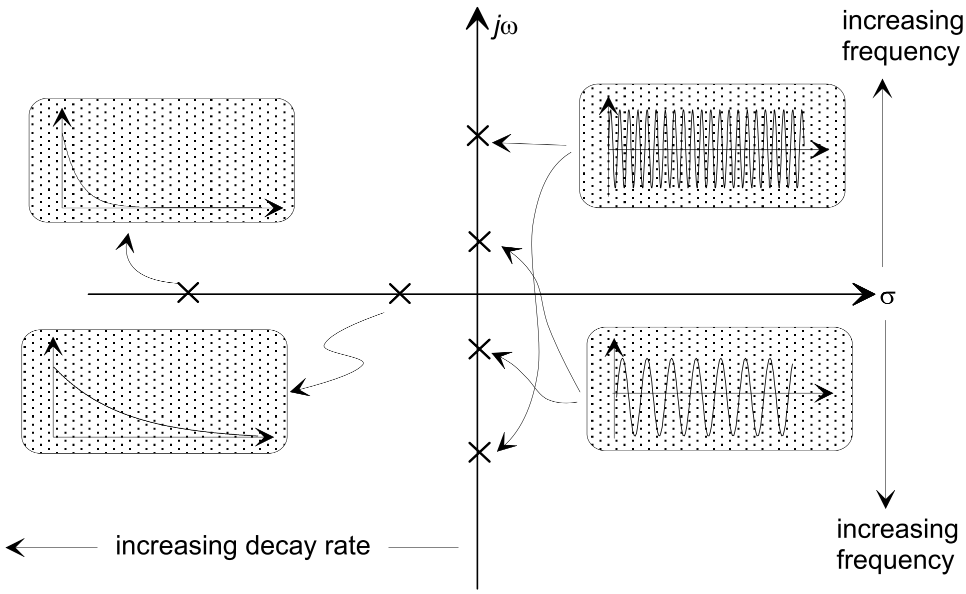
(c) Poles that lie on the imaginary axis ($\sigma = 0$) generate components that are purely oscillatory, and neither grow nor decay with time.

(d) A pole at the origin of the s-plane ($s = 0+j0$), generates a component that is a constant.

In addition we note that oscillatory frequency and decay rate is determined by the distance of the pole(s) from the origin.

(e) The rate of decay/growth is determined by the real part of the pole $\sigma = \Re\{s\}$, and poles deep in the lhp generate rapidly decaying components.

(f) For complex conjugate pole pairs, the oscillatory frequency is determined by the imaginary part of the pole pair $\omega = \Im\{s\}$.



1.3 System Stability

A system is defined to be *unstable* if its response from any finite initial conditions increases without bound. Since

$$Y_h(t) = \sum_{i=1}^{n} C_i e^{p_i t}$$

a system will be unstable if any component of $y_h(t)$ increases without bound, leading to the following statements:

> **(a)** A system is unstable if any pole has a positive real part (ie lies in the rhp), or equivalently
> **(b)** For a system to be stable, all poles must lie in the lhp.

A system with poles on the imaginary axis (with no poles in the rhp) is defined to be *marginally stable* since its homogeneous response from arbitrary initial conditions will neither decay to zero or increase to infinity.