2.004 Dynamics and Control II
Spring 2008

Massachusetts Institute of Technology

Department of Mechanical Engineering

2.004 *Dynamics and Control II*

Spring Term 2008

Lecture 20[1]

Reading:

- Nise: Secs. 4.1 – 4.6 (pp. 153 - 177)

1 Standard Forms for First- and Second-Order Systems

These are (a) all pole system (with no zeros), and (b) have unity gain ($\lim_{t\to\infty} y_{step}(t) = 1$).

1.1 First-Order System:

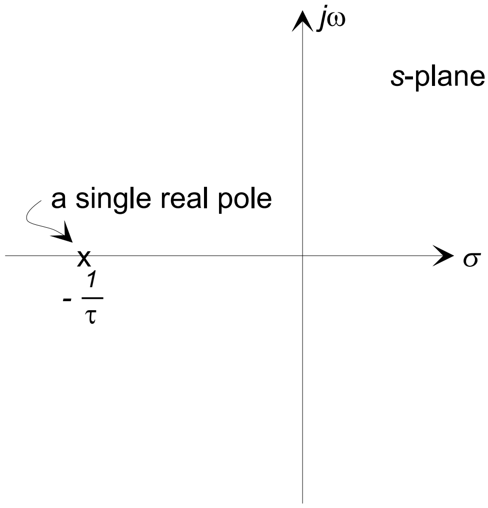
We define the first-order standard form as

$$\boxed{G(s) = \frac{1}{\tau s + 1},}$$

where the single parameter τ is the time constant. As a differential equation

$$\boxed{\tau \frac{dy}{dt} + y = u(t).}$$

and the system has a single real pole at $s = -1/\tau$.



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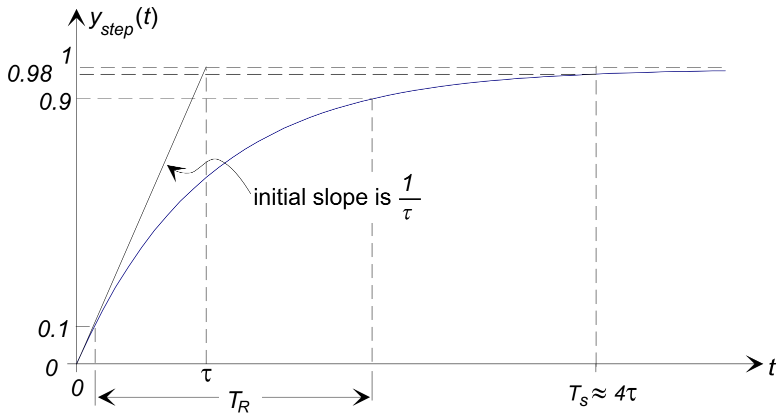
The step response is

$$y_{step} = \mathcal{L}^{-1}\left\{\frac{1}{s}\frac{1/\tau}{s+1/\tau}\right\} = 1 - e^{-t/\tau}$$



1.1.1 Common Step Response Descriptors:

(a) Settling Time: The time taken for the response to reach 98% of its final value. Since

$$y_{step}(t) = 1 - e{-}t/\tau$$

and $e^{-4} = 0.0183 \approx 0.02$, we take

$$\boxed{T_s = 4\tau}$$

as the definition of T_s.

(b) Rise Time: Commonly taken as the time taken for the step response to rise from 10% to 90% of the steady-state response to a step input. It is found from the step response as follows

$$0.1 = 1 - e^{-t_{0.1}/\tau} \quad \Rightarrow \quad t_{0.1} = \tau \ln(0.9)$$
$$0.9 = 1 - e^{-t_{0.9}/\tau} \quad \Rightarrow \quad t_{0.9} = \tau \ln(0.1)$$

$$\boxed{T_R = t_{0.9} - t_{0.1} = (\ln(0.1) - \ln(0.9))\tau = 2.2\tau}$$

1.2 Second-Order Systems

The standard unity gain second-order system has a transfer function

$$\boxed{G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$$

with two parameters (1) ω_n – the undamped natural frequency, and (ii) ζ – the damping ratio ($\zeta \geq 0$). The system poles are the roots of $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$, that is

$$\boxed{p_1, p_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1},}$$

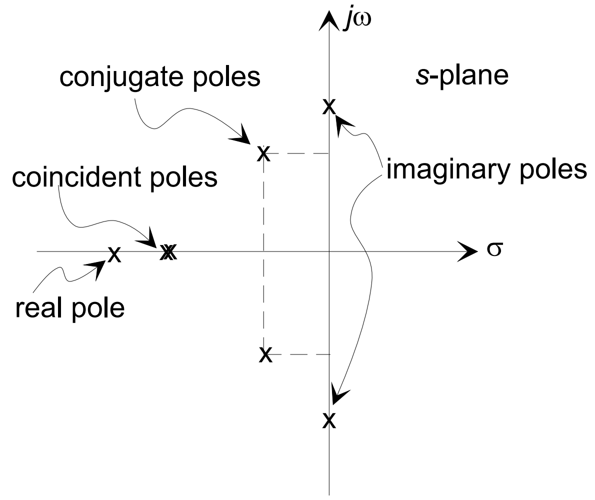
leading to four cases

i) $\zeta > 1$ – the poles are real and distinct
$p_1, p_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$,

ii) $\zeta = 1$ – the poles are real and coincident
$p_1, p_2 = -\zeta\omega_n$,

iii) $0 < \zeta < 1$ – the poles are complex conjugates
$p_1, p_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$, or

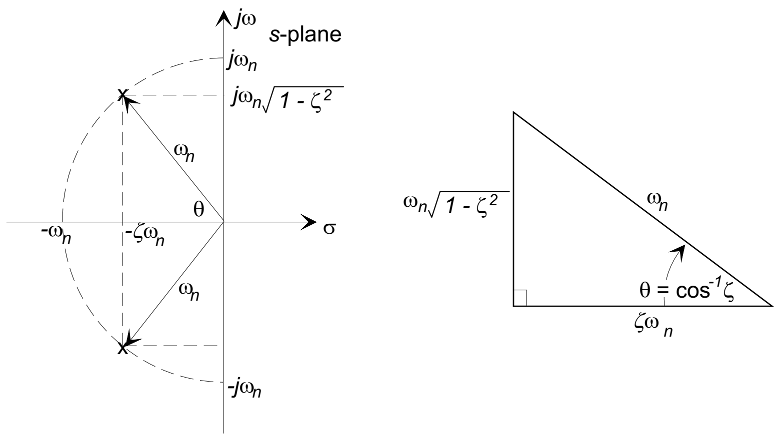
(iv) $\zeta = 0$ – the poles are purely imaginary
$p_1, p_2 = \pm j\omega_n$.



1.2.1 Pole Positions For an Underdamped Second-Order System

$$p_1, p_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$$

and when plotted on the s-plane



we note that

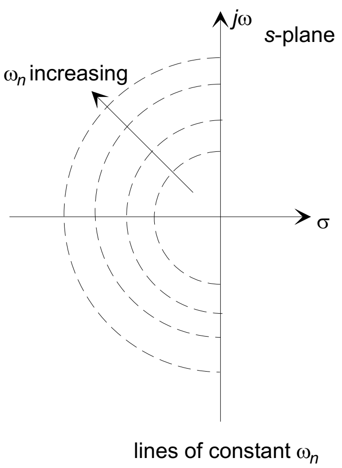
(a) The poles lie at a distance ω_n from the origin, and

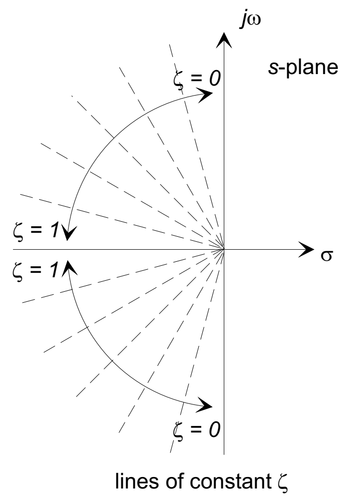
(b) The poles lie on radial lines at an angle

$$\theta = \cos^{-1}(\zeta)$$

as shown above.

The influence of ζ and ω_n on the pole locations may therefore be summarized:



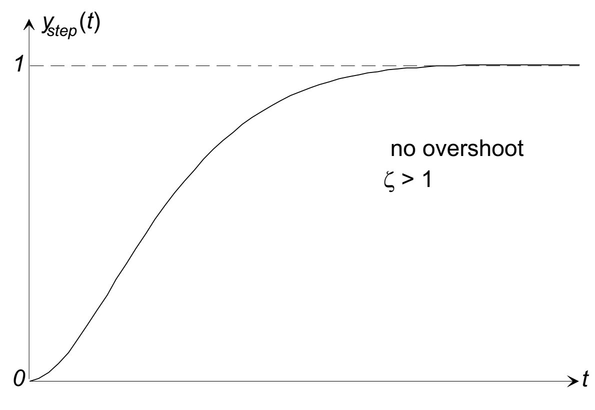


1.2.2 Step Responses

(a) The over damped case $(\zeta > 1)$

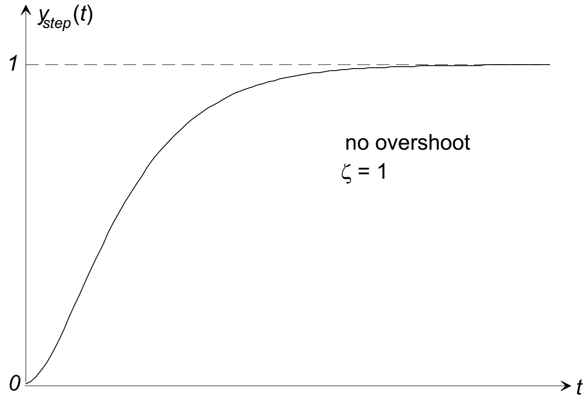
$$y_{step}(t) = 1 - C_1 e^{-p_1 t} - C_2 e^{-p_2 t}$$

where the constants C_1 and C_2 are determined from p_1 and p_2.



(b) The critically damped case $(\zeta = 1)$ With two coincident poles $p_1 = p_2 = p$, the step response takes a special form

$$y_{step}(t) = 1 - C_1 e^{pt} - C_2 t e^{-pt}$$



(c) The under damped case ($0 < \zeta < 1$) With a pair of complex conjugate poles

$$p_1, p_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

the step response becomes oscillatory

$$\boxed{y_{step}(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\cos\left(\omega_n\sqrt{1-\zeta^2}t - \phi\right)\right)}$$

where

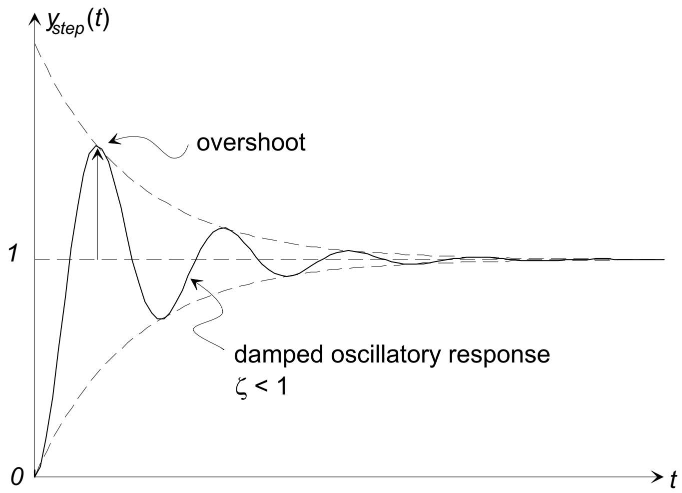
$$\phi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$$

and if we define the *damped natural frequency* ω_d as

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

we can write the step response as

$$\boxed{y_{step}(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\cos\left(\omega_d t - \phi\right)\right)}$$

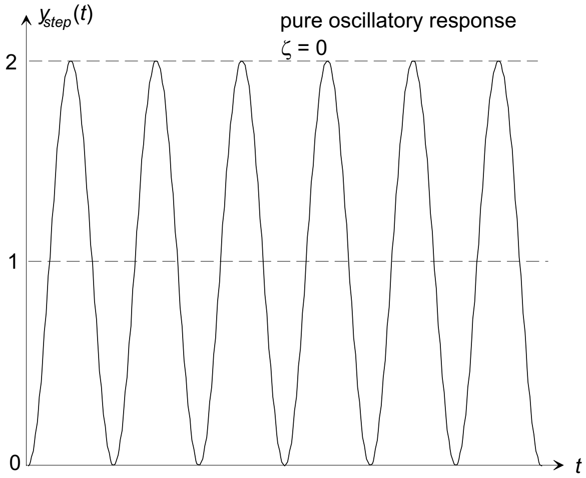


(d) The undamped case ($\zeta = 0$) In this case

$$G(s) = \frac{\omega_n}{s^2 + \omega_n^2}$$

and the poles are $p_1, p_2 = \pm j\omega_n$. Then

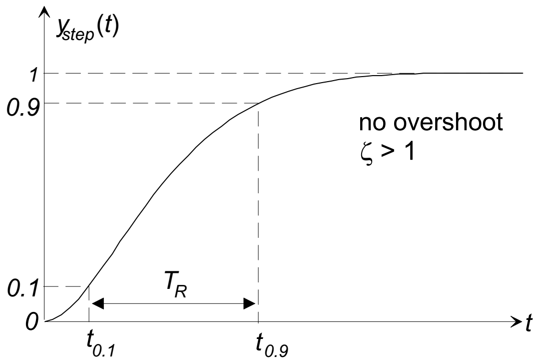
$$\boxed{y_{step}(t) = 1 - \cos(\omega_n t)}$$

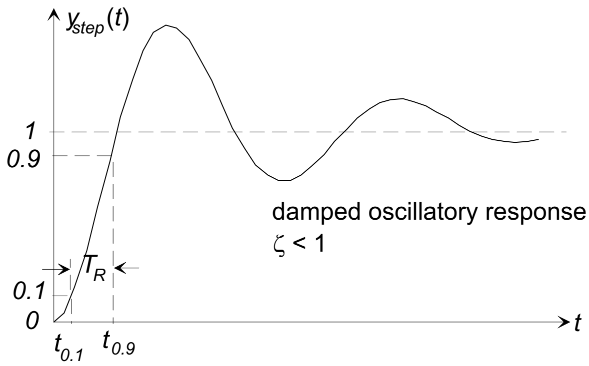


Note: For any second-order system, the initial slope of the step response is zero, since by definition the system is at rest at time $t = 0$, that is $y_{step}(0) = 0$, and $\dot{y}_{step}(0) = 0$.

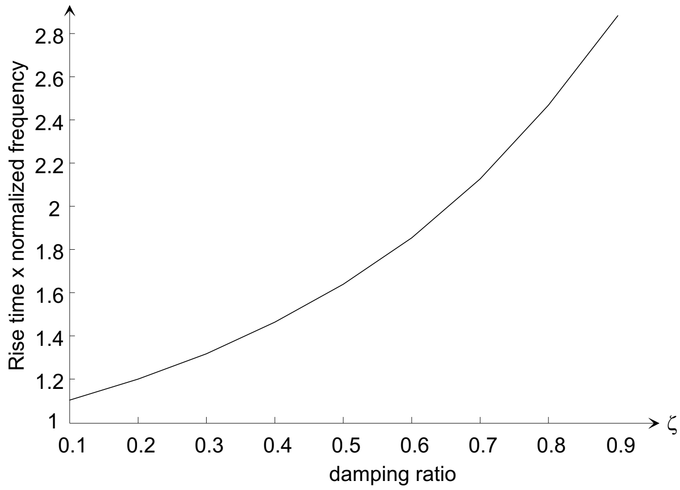
1.2.3 Step Response Based Second-Order System Specifications

(a) Rise Time (T_R): Applies to over- and under-damped systems. As in the case of first-order systems, the usual definition is the time taken for the step response to rise from 10% to 90% of the final value:

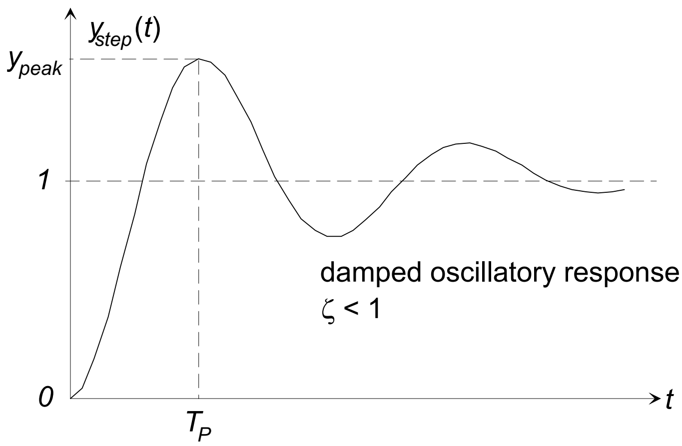




For a second-order system there is no simple (general) expression for T_R. The following figure - from Nise, Fig. 4.16, (p. 172) is derived empirically:



(b) Peak Time (T_p): Applies only to under-damped systems, and is defined as the time to reach the first peak of the oscillatory step response.



T_p is found by differentiating the step response $y_{step}(t)$, and equating to zero. (See Nise p. 170 for details.)

$$\boxed{T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}}$$

**** Transient response specifications continued in Lecture 21. ****