2.004 Dynamics and Control II
Spring 2008

Massachusetts Institute of Technology

Department of Mechanical Engineering

2.004 *Dynamics and Control II*

Spring Term 2008

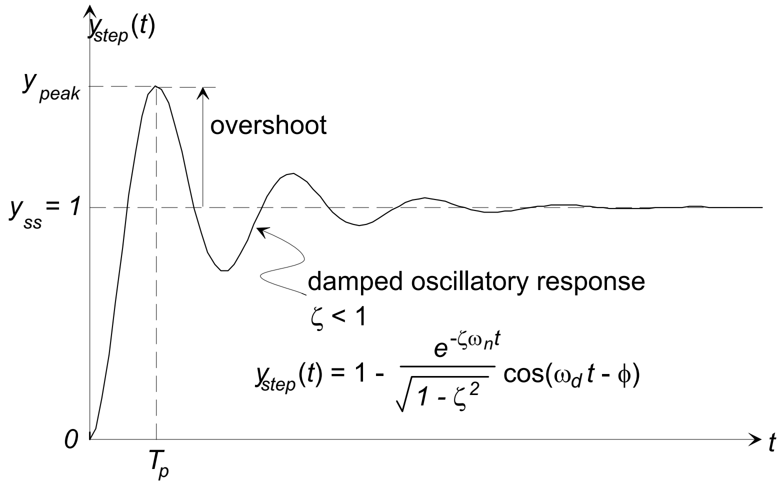
Lecture 21[1]

Reading:

- Nise: Secs. 4.6 – 4.8 (pp. 168 - 186)

1 Second-Order System Response Characteristics (contd.)

1.1 Percent Overshoot



The height of the first peak of the response, expressed as a percentage of the steady-state response.

$$\%\text{OS} = \frac{y_{peak} - y_{ss}}{y_{ss}} \times 100$$

At the time of the peak $y(T_p)$

$$y_{peak} = y(T_p) = 1 + e^{-(\zeta\pi/\sqrt{1-\zeta^2})}$$

and since $y_{ss} = 1$

$$\boxed{\%\text{OS} = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100.}$$

Note that the percent overshoot depends only on ζ.

Conversely we can find ζ to give a specific percent overshoot from the above:

$$\boxed{\zeta = \frac{-\ln(\%\text{OS}/100)}{\sqrt{\pi^2 + \ln^2(\%\text{OS}/100)}}}$$

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■ Example 1

Find the damping ratio ζ that will generate a 5% overshoot in the step response of a second-order system.

Using the above formula

$$\zeta = \frac{-\ln\left(\%OS/100\right)}{\sqrt{\pi^2 + \ln^2\left(\%OS/100\right)}} = \frac{-\ln(0.05)}{\sqrt{\pi^2 + \ln^2(0.05)}} = 0.69$$

■ Example 2

Find the location of the poles of a second-order system with a damping ratio $\zeta = 0.707$, and find the corresponding overshoot.

The complex conjugate poles line on a pair of radial lines at an angle

$$\theta = \cos^{-1} 0.707 = 45^\circ$$

from the negative real axis. The percentage overshoot is

$$\begin{aligned} \%OS &= e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100 \\ &= e^{-(0.707\pi/\sqrt{1-0.5})} \times 100 \\ &= 4.3\% \quad (\approx 5\%) \end{aligned}$$

The value $\zeta = .707 = \sqrt{2}/2$ is a commonly used specification for system design and represents a compromise between overshoot and rise time.

1.2 Settling Time

The most common definition for the settling time T_s is the time for the step response $y_{step}(t)$ to reach and stay within 2% of the steady-state value y_{ss}. A conservative estimate can be found from the decay envelope, that is by finding the time for the envelope to decay to less than 2% of its initial value,

$$\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} < 0.02$$

giving

$$\boxed{T_s = -\frac{\ln(0.02\sqrt{1-\zeta^2})}{\zeta\omega_n}}$$

or

$$\boxed{T_s \approx \frac{4}{\zeta\omega_n} \quad \text{for} \quad \zeta^2 \ll 1.}$$

■ Example 3

Find (i) the pole locations for a system under feedback control that has a peak time $T_p = 0.5$ sec, and a 5% overshoot. Find the settling time T_s for this system.

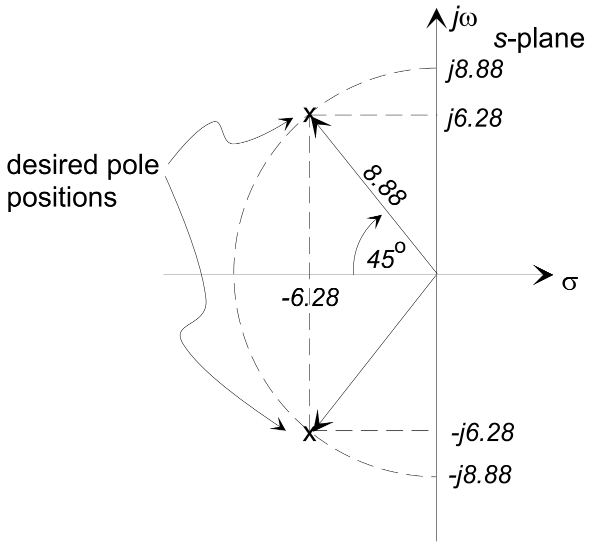
From Example 2 we take the desired damping ratio $\zeta = 0.707$. Then

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.5 \text{ s}$$

so that

$$\omega_n = \frac{\pi}{T_p\sqrt{1-\zeta^2}} = \frac{\pi}{0.5\sqrt{1-0.5}} = 8.88\text{rad/s}.$$

The pole locations are shown below:



Then

$$\begin{aligned} p_1, p_2 &= -8.88\cos\left(\frac{\pi}{4}\right) \pm j8.88\sin\left(\frac{\pi}{4}\right) \\ &= -6.28 \pm j6.28 \end{aligned}$$

The indicated settling time T_s from the approximate formula is

$$T_s \approx \frac{4}{\zeta\omega_n} = \frac{4}{0.707 \times 8.88} = 0.64 \text{ s}.$$

Note that in this case ζ does not meet the criterion $\zeta^2 \ll 1$ and the full expression

$$T_s = -\frac{\ln(.02\sqrt{1-\zeta^2})}{\zeta\omega_n} = -\frac{\ln(.02\sqrt{1-0.5})}{0.5 \times 8.88} = 0.68 \text{ s}$$

gives a slightly larger value.

2 Higher Order Systems

For systems with three or more poles, the system be analyzed as a parallel combination of first- and second-order blocks, where complex conjugate poles are combined into a single second-order block with real coefficients, using partial fractions. The total system output is then the superposition of the individual blocks.

■ Example 4

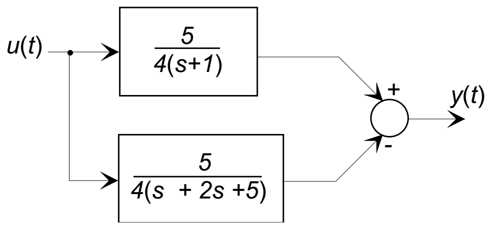
Express the system

$$G(s) = \frac{5}{(s+1)(s^2+2s+5)}$$

as a parallel combination of first- and second-order blocks.

$$\begin{aligned} G(s) &= \frac{5}{(s+1)(s^2+2s+5)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+2s+5} \\ &= \frac{5}{4}\frac{1}{s+1} - \frac{5}{4}\frac{s+1}{s^2+2s+5} \end{aligned}$$

using partial fractions. The system is described by the following block diagram



and the response to an input $u(t)$ may be found as the (signed) sum of the responses of the two blocks.

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3 Some Fundamental Properties of Linear Systems

3.1 The Principle of Superposition

For a linear system at rest at time $t = 0$, if the response to an input $u(t) = f(t)$ is $y_f(t)$, and the response to a second input $u(t) = g(t)$ is $y_g(t)$, then the response to an input that is a linear combination of $f(t)$ and $g(t)$, that is

$$u(t) = af(t) + bg(t)$$

where a and b are constants is

$$y(t) = ay_f(t) + by_g(t).$$

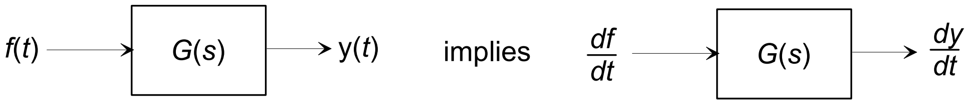
3.2 The Derivative Property

For a linear system at rest at time $t = 0$, if the response to an input $u(t) = f(t)$ is $y_f(t)$, then the response to an input that is the derivative of $f(t)$, that is

$$u(t) = \frac{df}{dt}$$

is

$$y(t) = \frac{dy_f}{dt}.$$



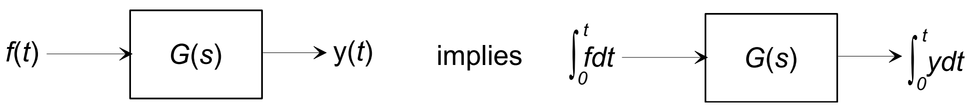
3.3 The Integral Property

For a linear system at rest at time $t = 0$, if the response to an input $u(t) = f(t)$ is $y_f(t)$, then the response to an input that is the integral of $f(t)$, that is

$$u(t) = \int_0^t f(t)dt$$

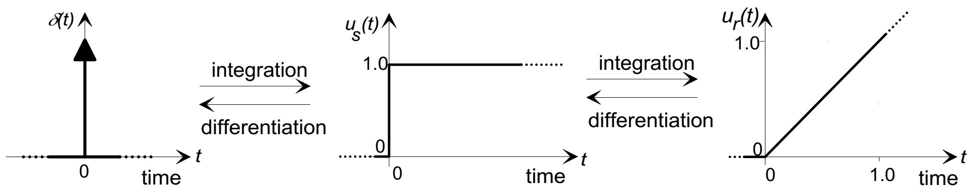
is

$$y(t) = \int_0^t y_f(t)dt.$$



■ Example 5

We can use the derivative and integral properties to find the impulse and ramp responses from the step response. We have seen



therefore

$$\begin{aligned} y_\delta(t) &= \frac{d}{dt} y_{step}(t) \\ y_r(t) &= \int_0^t y_{step}(t)dt \end{aligned}$$

For example, consider

$$G(s) = \frac{b}{s+a}$$

with step response

$$y_{step} = \frac{b}{a}\left(1 - e^{-at}\right).$$

The impulse response is

$$y_\delta(t) = \frac{d}{dt} y_{step}(t) = be^{-at},$$

and the ramp response is

$$y_r(t) = \int_0^t y_{step}(t)dt = \frac{b}{a}\left(t + \frac{1}{a}\left(1 - e^{-at}\right)\right)$$

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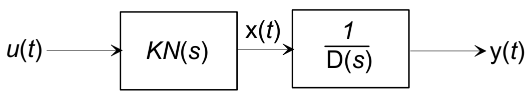
4 The Effect of Zeros on the System Response

Consider a system with a transfer function:

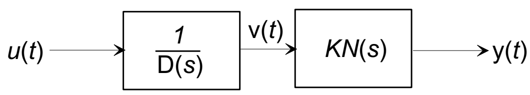
$$G(s) = K\frac{N(s)}{D(s)} = K\frac{s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}.$$

$N(s)$, which defines the system zeros, is associated with the RHS of the differential equation, while $D(s)$ is derived from the LHS of the differential equation. Therefore $N(s)$ does not affect the homogeneous response of the system.

We can draw $G(s)$ as cascaded blocks in two forms:



or



In this case we consider the all-pole system $1/D(s)$ to be excited by $x(t)$, which is a superposition of the derivatives of $u(t)$

$$x(t) = K\sum_{k=0}^{m} b_k \frac{d^k u}{dt^k}$$

In this case we consider the all-pole system $1/D(s)$ to be excited by $u(t)$ directly to generate $x(t)$, and the output is formed as a superposition of the derivatives of $v(t)$

$$y(t) = K\sum_{k=0}^{m} b_k \frac{d^k v}{dt^k}$$

■ Example 6

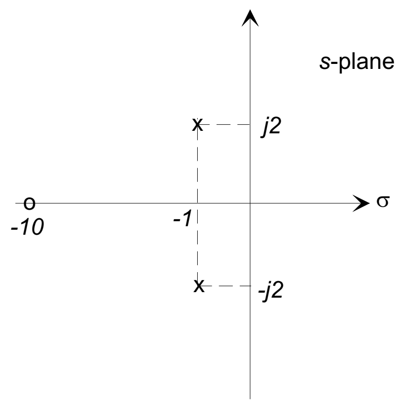
Find the step response of

$$G(s) = \frac{s+10}{s^2+2s+5}.$$

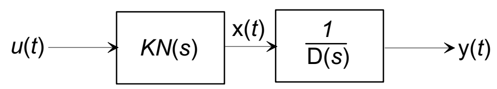
Splitting up the transfer function

$$N(s) = s+10, \qquad D(s) = \frac{1}{s^2+2s+5}$$

and $\omega_n = \sqrt{5}$, and $\zeta = 1/\sqrt{5}$.



Method 1: For the case,



if $u(t) = u_s(t)$, the unit-step function

$$x(t) = \frac{du}{dt} + 10u = \delta(t) + 10u_s(t)$$

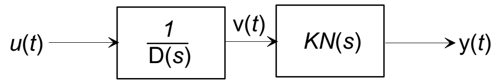
and for the all-pole system $1/D(s)$

$$\begin{aligned} y_{step}(t) &= \frac{1}{5}\left(1 - e^{-t}\cos(2t) - e^{-t}\frac{1}{2}\sin(2t)\right) \\ y_\delta(t) &= \frac{1}{2}e^{-t}\sin(2t). \end{aligned}$$

For the complete system

$$\begin{aligned} y(t) &= y_\delta(t) + 10y_{step}(t) \\ &= 2 - 2e^{-t}\cos(2t) - \frac{1}{2}e^{-t}\sin(2t) \end{aligned}$$

Method 2: For the case,



from above the step-response to the all-pole system $1/D(s)$ is

$$v(t) = \frac{1}{5}\left(1 - e^{-t}\cos(2t) - \frac{1}{2}e^{-t}\sin(2t)\right)$$

and the system output is

$$\begin{aligned}
y(t) &= \frac{dv}{dt} + 10v \\
&= \frac{1}{2}e^{-t}\sin(2t) + \frac{10}{5}\left(1 - e^{-t}\cos(2t) - \frac{1}{2}\sin(2t)\right) \\
&= 2 - 2e^{-t}\cos(2t) - \frac{1}{2}e^{-t}\sin(2t)
\end{aligned}$$

which is the same as found in Method 1.