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2.004 Dynamics and Control II
Spring 2008

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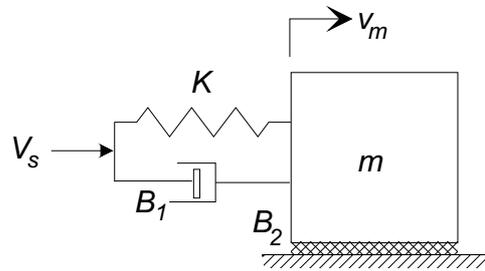
Lecture 23¹

Reading:

- Nise: 4.7, 4.8

1 Pole/Zero Cancellation:

Consider the following mechanical system:



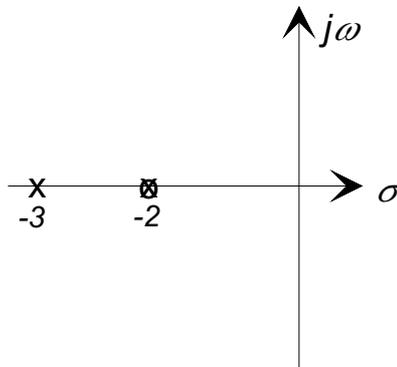
The transfer function relating v_m to V_s is

$$G(s) = \frac{v_m(s)}{V_s(s)} = \frac{B_1 s + K}{m s^2 + (B_1 + B_2) s + K}$$

which is clearly second-order. However, if the parameter values are $m_1 = 1/3$ kg, $K = 2$ N/m, $B_1 = 1$ N-s/m, $B_2 = 2/3$ N-s/m the transfer function becomes

$$G(s) = \frac{s + 2}{(1/3)s^2 + (5/3)s + 2} = \frac{3(s + 2)}{(s + 2)(s + 3)}$$

with a coincident pole and zero at $s = -2$.



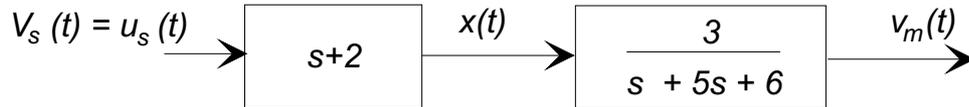
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Clearly, with these values, cancelation takes place in the transfer function and

$$G(s) = \frac{3}{s + 3}$$

This phenomenon is known as *pole/zero cancelation*. With these values the response will be identical to a first-order system with a pole at $s = -3$.

Let's compute the step response of this system to understand what has happened. We start by breaking the system into two cascaded blocks as we have discussed previously:



When the input $V_s(t) = u_s(t)$, the unit-step (Heaviside) function, the intermediate variable $x(t)$ is

$$x(t) = \frac{du_s(t)}{dt} + 2u_s(t) = \delta(t) + 2u_s(t)$$

where $\delta(t)$ is the Dirac delta function. The system output $v_m(t)$ will therefore be

$$v_m(t) = y_\delta(t) + 2y_s(t)$$

where $y_\delta(t)$ is the *impulse response* of the all-pole system to the right, and $y_s(t)$ is its *step response*. You can show for yourself that

$$\begin{aligned} y_s(t) &= 0.5 - 1.5e^{-2t} + e^{-3t} \\ y_\delta(t) &= 3e^{-2t} - 3e^{-3t} \end{aligned}$$

so that

$$\begin{aligned} v_m(t) &= y_\delta(t) + 2y_s(t) = (3e^{-2t} - 3e^{-3t}) + 2(0.5 - 1.5e^{-2t} + e^{-3t}) \\ &= 1 - e^{-3t} \end{aligned}$$

which is identical to the step response of the reduced system

$$G(s) = \frac{3}{s + 3}$$

Notice that the modal component e^{-2t} has been canceled from the response. Although we have simply demonstrated the cancelation in this example, the following statement is true for all inputs:

For any system in which one or more poles have been canceled by zeros in the transfer function, the modal components e^{pt} corresponding to the canceled poles will not appear in the output for any input $u(t)$.

We say that the canceled modes are not *excited* by the input.

2 The Effect of Zeros in the Proximity of a Pole

The previous discussion considered only the complete cancelation of a pole. In many systems there may be zeros close to a pole but not coincident with it. Let's address this issue by example again by considering the step response of a second-order system with two real poles and a single, this time in symbolic form:

$$G(s) = \frac{ab}{c} \frac{(s+c)}{(s+a)(s+b)}$$

where the constant ab/c forces this to be a unity-gain system.

The step-response for this system is:

$$y_s(t) = 1 - \frac{b(a-c)}{c(a-b)} e^{-at} + \frac{a(b-c)}{c(a-b)} e^{-bt}.$$

Note that the amplitude of each of the modal components is determined by the *distance from the pole to the zero*, that is $(a-c)$ for the component e^{-at} , and $(b-c)$ for the component e^{-bt} , and although this is simply a demonstration The following result is true:

As a zero approaches a pole, the amplitude of the modal component e^{pt} corresponding to the pole decreases, for any input $u(t)$.

This example also demonstrates the complete elimination of the component e^{pt} when the zero and pole are coincident.

■ Example 1

Compare the step responses of the system

$$G_1(s) = \frac{(12/c)(s+c)}{(s+3)(s+4)}$$

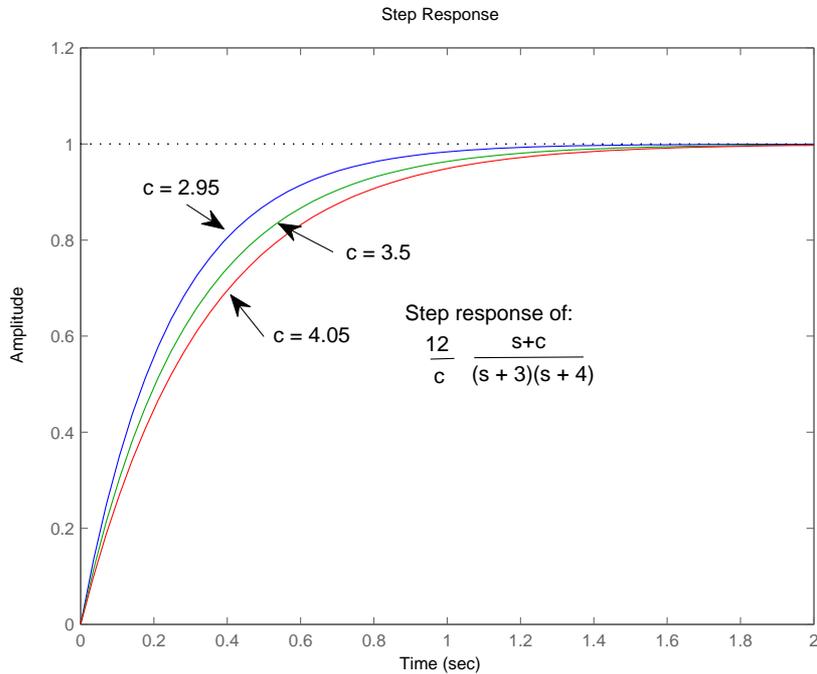
for

- (i) $c = 2.95$ (ie, a zero close to the pole at $s = -3$).
- (ii) $c = 3.50$ (ie, a zero mid-way between the two poles).
- (iii) $c = 4.05$ (ie a zero close to the pole at $s = -4$).

Substitution into the above gives the three step responses as

$$\begin{aligned} c = 2.95: & \quad y_s(t) = 1 + 0.068e^{-3t} - 1.068e^{-4t} \\ c = 3.50: & \quad y_s(t) = 1 - 0.571e^{-3t} - 0.429e^{-4t} \\ c = 4.05: & \quad y_s(t) = 1 - 1.007e^{-3t} + 0.007e^{-4t} \end{aligned}$$

The effect on the amplitude of the component associated with the pole close to the zero is obvious. The three responses are plotted below:



3 The Response of High-Order Systems

The exact form of the response of third-order and higher systems is not easy to tabulate and summarize. For example, a third-order system may have three real poles, or two complex conjugate poles and a single real pole. The situation becomes even more complicated for fourth- and higher order systems. Nevertheless the same basic rules apply as we have developed for first- and second-order systems:

- The total response to an input $u(t)$ will consist of two fundamental components: the *homogeneous response* $y_h(t)$, and a *particular solution* $y_p(t)$ that is defined by $u(t)$.
- The homogeneous component will consist a set of a superposition of n modal components of the form e^{pt} associated with the system poles, that is

$$y_h(t) = \sum_{i=1}^n C_i e^{p_i t}$$

where the constants C_i are later determined by the assumption that the system is at rest at $t = 0$.

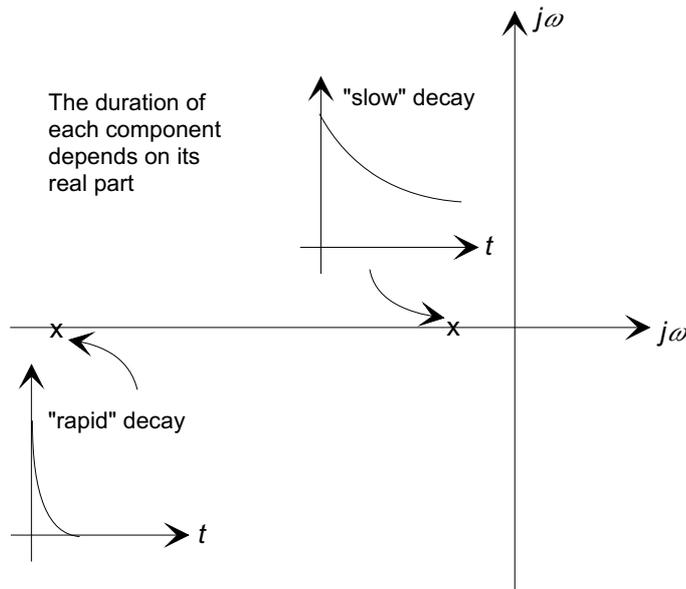
- The particular solution is usually determined (if necessary) by the method of undetermined coefficients, in which it is assumed that $y_p(t)$ is similar in form to the input. The value of $y_p(t)$ is affected by the system zeros.
- The total solution is therefore

$$y(t) = \sum_{i=1}^n C_i e^{p_i t} + y_p(t)$$

The pole-zero plot essentially tells us what the components of the response will be, but gives little information about the strength of the components. Never the less it is a very useful tool in control system design.

4 Model Approximation – Dominant Poles:

We have seen that the modal component of a real pole at $s = \sigma$ is $Ce^{\sigma t}$, and the component due to a complex conjugate pole pair at $s = \sigma \pm j\omega$ is $Ce^{\sigma t} \cos(\omega t + \phi)$, and that the decay rate of the component is defined by the real part of the pole. Poles lying deep in the l.h. plane decay much faster than those close to the imaginary axis.



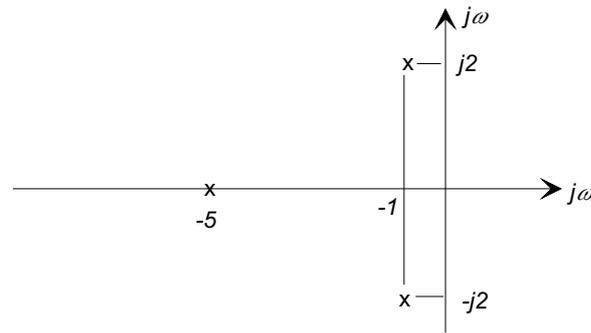
Poles lying close to the imaginary axis are known as *dominant poles* because their response components persist for much longer than those deep within the l.h. plane.

There is often the need to simplify a system model, for example to approximate a third-order system by a second-order model. This can sometimes be done by retaining only the dominant pole pair - provided the third pole is sufficiently well separated from the dominant pair.

Consider a system shown with transfer function

$$G(s) = \frac{50}{s^3 + 12s^2 + 255s + 50} = \frac{50}{(s + 10)(s^2 + 2s + 5)}$$

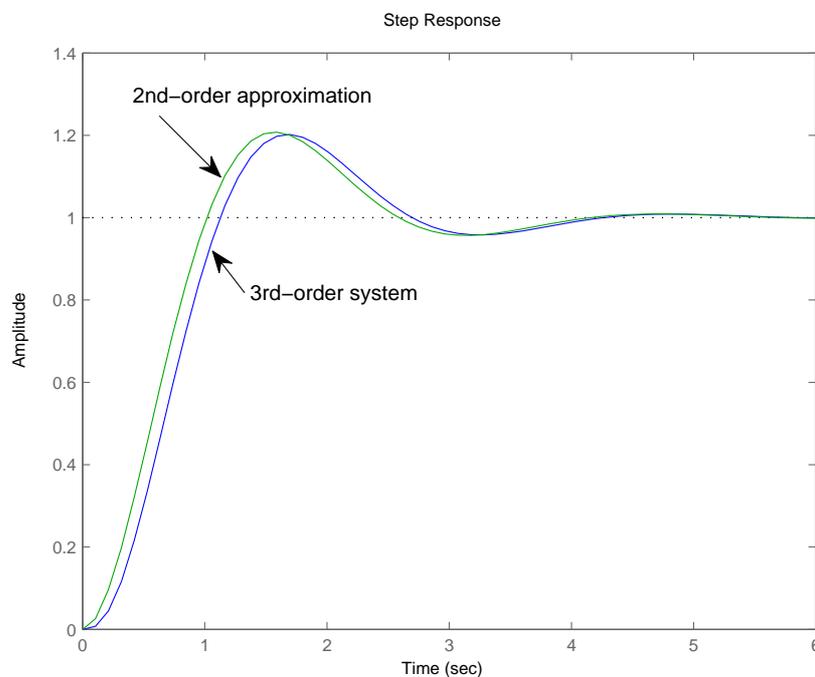
which has poles at $s = -10, -1 \pm j2$, as shown below:



By ignoring the pole at $s = -10$, and maintaining a unity-gain system we might assert

$$G(s) \approx \frac{5}{s^2 + 2s + 5}$$

The following figure compares the step response of the two systems.



It can be seen that the responses are similar.

Note: There is a danger in using reduced-order models in closed-loop control system design. For example the use of a second-order approximation to a real third-order system will indicate that the system will never become unstable with proportional control. The physical system, however, will become unstable as the proportional gain is increased.

For example, the system shown in the above example will be unstable under proportional control for $K_p > 5$, which cannot be predicted from the second-order approximation.