2.004 Dynamics and Control II
Spring 2008

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

DEPARTMENT OF MECHANICAL ENGINEERING

2.004 *Dynamics and Control II*

Spring Term 2008

Lecture 26[1]

Reading:

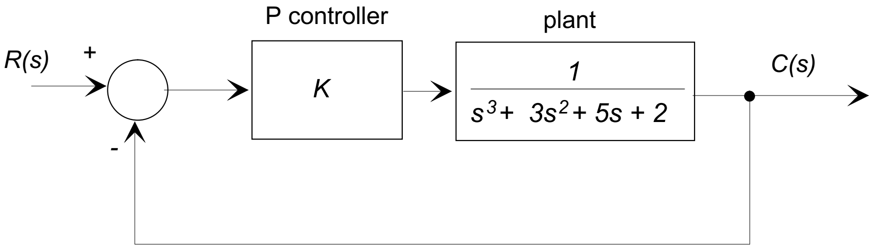
- Nise: Chapter 6
- Nise: Chapter 8

1 Determining Stability Bounds in Closed-Loop Systems

Consider the closed-loop third-order system with proportional controller gain K with open-loop transfer function

$$G_f(s) = \frac{K}{s^3 + 3s^2 + 5s + 2}$$

shown below.



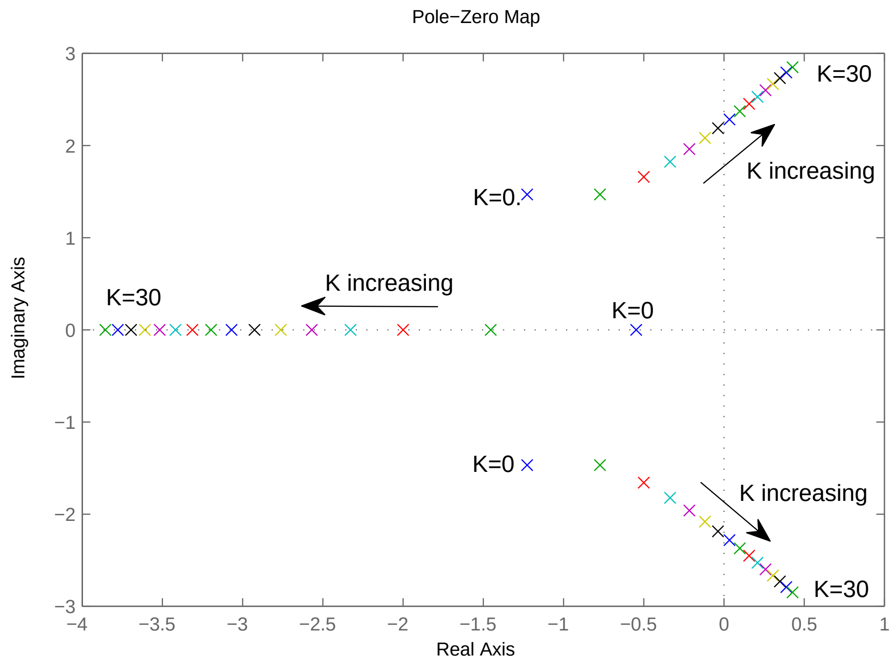
The closed loop transfer function is:

$$G_{cl}(s) = \frac{N(s)}{D(s) + N(s)} = \frac{K}{s^3 + 3s^2 + 5s + (2 + K)}$$

Let's examine the closed-loop stability by using the `pzmap()` function in MATLAB:

```
 sys = tf(1,[1 3 5 2]);
pzmap(sys);
hold on;
for K = 2:2:30
    sys = tf(K,[1 3 5 2+K]);
    pzmap(sys);
end;
```

which superimposes the closed-loop pole/zero plots for $K = 0 \ldots 30$ on a single plot:



From the plot we note the following:

- This system always has two complex conjugate poles and a single real pole.
- When $K = 0$ the poles are the open-loop poles.
- As K increases, the real pole moves deeper into the l.h. plane, and the complex conjugate poles approach and cross the imaginary $(j\omega)$ axis, and the system becomes unstable.
- Close examination of the plot shows that the system becomes unstable at a value of K between $K = 12$ and $K = 14$.

We now look at three methods for determining the stability limit of the proportional gain K for this system.

■ Example 1

Use the Routh-Hurwitz method to find the range of proportional controller gain K for which the above system will be stable.

The first two rows of the Routh array are taken directly from $D(s)$:

s^3	1	5	0
s^2	3	$2+K$	0

and the next two rows are computed as above

$$\begin{aligned} b_1 &= -\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} = -\frac{1}{3}\begin{vmatrix} 1 & 5 \\ 3 & 2+K \end{vmatrix} = -\frac{1}{3}(K-13) \\ b_2 &= -\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix} = -\frac{1}{3}\begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} = 0 \end{aligned}$$

Similarly, the s^0 row is computed

$$\begin{aligned} c_1 &= -\frac{1}{b_1}\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix} = -\frac{3}{K-13}\begin{vmatrix} 3 & 24 \\ -(K-13)/3 & 2+K \end{vmatrix} = 2+K \\ c_2 &= -\frac{1}{b_1}\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_3 \end{vmatrix} = -\frac{3}{K-13}\begin{vmatrix} 3 & 0 \\ -(K-13)/3 & 0 \end{vmatrix} = 0 \end{aligned}$$

and the complete Routh array is

s^3	1	5	0
s^2	3	2+K	0
s^1	-(K-13)/3	0	
s^0	(2+K)	0	

We now examine the first column to determine the range of proportional gain for which this system will be stable. In order for there to be no sign changes we require

$$-2 < K < 13$$

We conclude that if $K < -2$ there will be one (therefore real) unstable pole, while if $K > 13$ there will be two unstable poles. When $K = 13$ the denominator is

$$D(s) = s^3 + 3s^2 + 5s + 15 = (s+3)(s+j2.236)(s-j2.236)$$

so that the closed-loop system has a pair of poles on the imaginary axis. The system will be marginally stable (a pure oscillator at a frequency of $\omega = 2.236$ r/s).

■ Example 2

Use the stability criterion for third-order systems developed in Example 3 of Lecture 25 to determine the stability bounds for the above system.

In Lecture 25 we showed that for a third-order system with characteristic equation:

$$D(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$$

the system is stable only if

$$a_1 a_2 > a_0 a_3$$

In this case

$$D(s) = s^3 + 3s^2 + 5s + (2 + K)$$

and therefore for stability we require

$$15 > 2 + K$$

or $K < 13$.

■ Example 3

Use the characteristic equation directly to find the closed-loop stability limits for the above system. There are three closed-poles. We conjecture that at the stability boundary (marginal stability) there will be a pair of poles on the imaginary axis at $s = \pm j\omega$, and a single real pole at $s = -a$.

The closed-loop characteristic polynomial will therefore be

$$D(s) = (s + a)(s^2 + \omega^2) = s^3 + as^2 + \omega^2 s + a\omega^2$$

Comparing coefficients with the actual closed-loop characteristic polynomial

$$D(s) = s^3 + 3s^2 + 5s + (2 + K)$$

we determine

$$\begin{aligned} a &= 3 \\ \omega^2 &= 5 \qquad \rightarrow \qquad \omega = \sqrt{5} \\ a\omega^2 &= K + 2 \qquad \rightarrow \qquad K = 13 \end{aligned}$$

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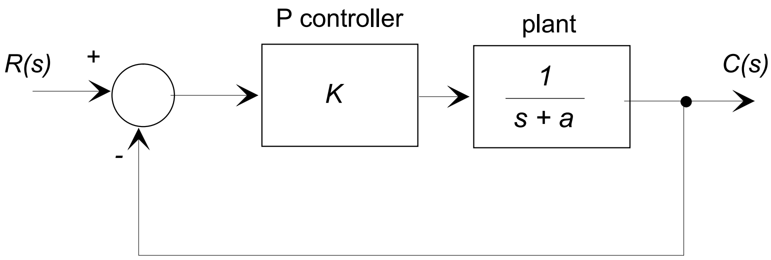
2 Root Locus Methods

We have seen that the closed-loop poles change as controller parameters vary. A *root-locus* is is an s-plane plot of the paths that the closed-loop poles take as a controller parameter varies. Let's start with some simple examples.

■ Example 4

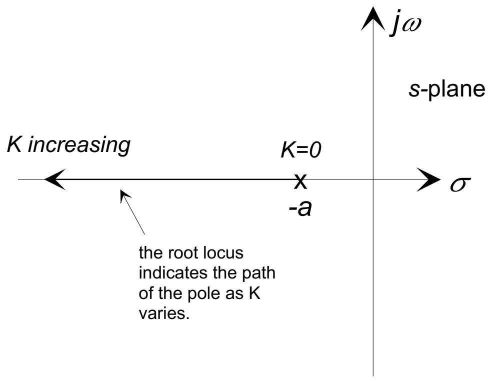
Consider the first order plant under proportional control, as shown below:



The closed-loop transfer function is

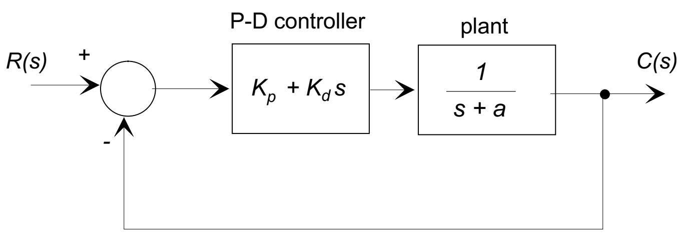
$$G_{cl}(s) = \frac{K}{s + (a + K)}$$

with a single pole $p_{c]} = -(a + K)$. The root-locus is simply the path of this pole as K varies from $K = 0$ to $K = \infty$. Clearly as $K \to 0$, the closed-loop pole approaches the open-loop pole ($s = -a$), and as $K \to \infty$, the closed-loop pole $p \to -\infty$. This is all the information we need to construct the root-locus for this system.



■ Example 5

Construct the root-locus plot for the first-order system under P-D control with $G_c(s) = K_p + K_d s$:



If we write

$$G_c(s) = K_d \left(s + \frac{K_p}{K_d} \right)$$

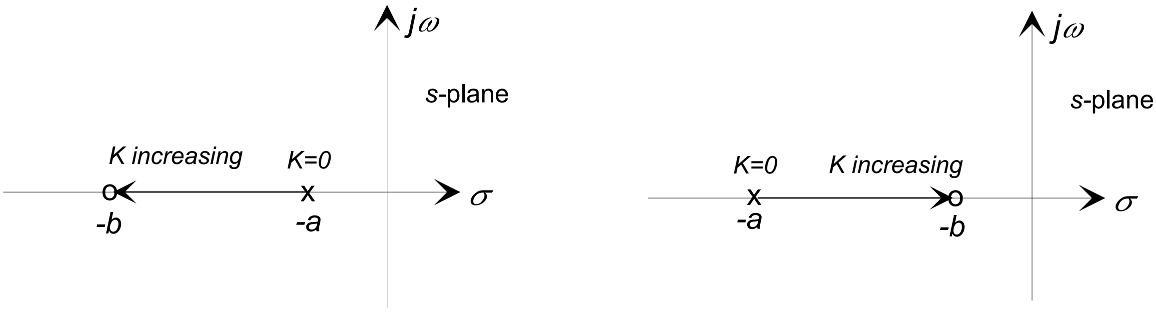
we have a open-loop pole at $s = -a$ and an open-loop zero at $s = -K_p/K_d = -b$. The closed-loop transfer function is

$$G_{cl}(s) = \frac{K_d(s+b)}{(K_d+1)s + (a + K_d b)}$$

with a single pole

$$p_{cl} = -\frac{a + bK_d}{1 + K_d}.$$

We now construct the root-locus as K_d varies from 0 to ∞. Clearly as $K_d \to 0$, the closed-loop pole $p_{cl} \to -a$ approaches the open-loop pole at $(s = -a)$, and as $K_d \to \infty$, the closed-loop pole $p_{cl} \to -b$, in other words the closed-loop pole approaches the open-loop zero. There are two possibilities for the root locus based on the relative positions of the open-loop pole and zero:



(a) $b > a$ (b) $a > b$

While the root locus always *originates* at the pole and *terminates* at the zero, if $b > a$ the closed-loop pole will move to the left, while if $a > b$ the pole will move to the right. In addition we can note:

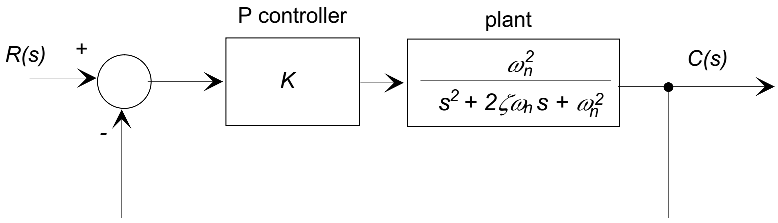
- There is a closed-loop zero at $s = -b$.
- This is a case when the order of the numerator is equal to the order of the denominator, and there will be direct feed-through from the input to the output, as discussed in Lecture 22.
- As K is increased, and the closed-loop pole approaches the zero, the strength of the component $e^{p_{cl}t}$ in the response will be diminished (Lecture 23).

■ Example 6

Determine the root locus for the second-order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

under proportional control.



The closed-loop transfer function is

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2(1+K)}$$

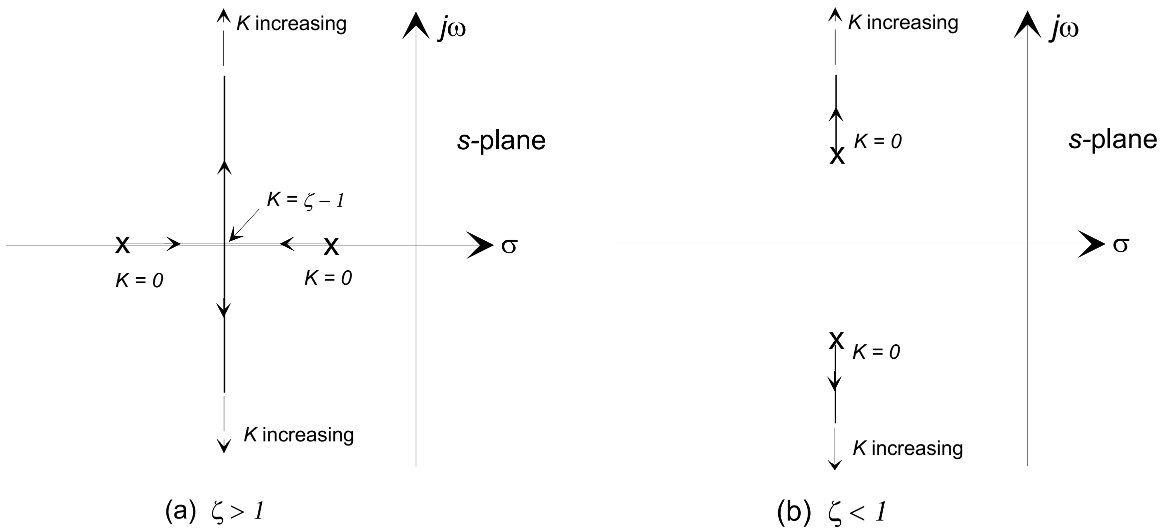
with closed-loop poles

$$p_1, p_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - (1+K)}$$

which will be real only if $\zeta \geq 1$ and $K \leq \zeta - 1$, otherwise they will be complex conjugates. We note the following:

- As $K \to 0$, the closed-loop poles approach the open-loop poles $-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$.
- If the open-loop poles are real, the closed-loop poles will move together as $K \to \zeta^2 - 1$, and then become complex.
- If the closed-loop poles are complex, $p_1, p_2 = -\zeta\omega_n \pm j\omega_n\sqrt{(1+K) - \zeta^2}$, only the imaginary part is affected by K, and as $K \to \infty$ the closed-loop poles $p_1, p_2 \to -\zeta\omega_n \pm j\infty$.

This behavior is summarized in the following root locus plots:



2.1 Some Basic Properties of Root Locus Plots

2.1.1 The Number of Branches in the Plot

By definition there will be one branch of the plot for each closed-loop pole. For a system with open-loop transfer function

$$G_{ol}(s) = K\frac{N_{ol}(s)}{D_{ol}(s)}$$

The closed-loop characteristic polynomial is

$$D_{cl}(s) = D_{ol}(s) + KN_{ol}(s)$$

and provided the order of $N_{ol}(s)$ does not exceed that of $D_{ol}(s)$, the order of $D_{cl}(s)$ will be the same as that of $D_{ol}(s)$. In other words, the number of closed-loop poles equals the number of open-loop poles, and the number of branches equals the number of open-loop poles..

2.1.2 Symmetry of the Root Locus Plot

Because all closed-loop poles are either real or complex conjugate pairs, the root locus is symmetrical about the real axis. The implication of this is that when we discuss rules for generating a root locus, we only have to consider half of the s-plane.

2.1.3 The Origins of the Branches ($K = 0$)

The closed-loop characteristic polynomial is

$$D_{cl}(s) = D_{ol}(s) + KN_{ol}(s).$$

As $K \to 0$, $D_{cl}(s) \approx D_{ol}(s)$, with the result that the n branches of the root locus *always originate at the open-loop poles.*

2.1.4 The Terminal Points of the Branches ($K \to \infty$)

As K becomes large

$$D_{cl}(s) \approx K N_{ol}(s)$$

with the result that m of the n closed-loop roots approach the m open-loop zeros. This leaves $n - m$ roots to be accounted for, and we will investigate this later. For now we simply state that these branches diverge away from the origin along a set of $n - m$ straight-line asymptotes, and as $K \to \infty$ these poles approach a distance $r = \infty$ from the origin.