2.004 Dynamics and Control II
Spring 2008

Massachusetts Institute of Technology
Department of Mechanical Engineering

2.004 *Dynamics and Control II*
Spring Term 2008

Lecture 30[1]

Reading:

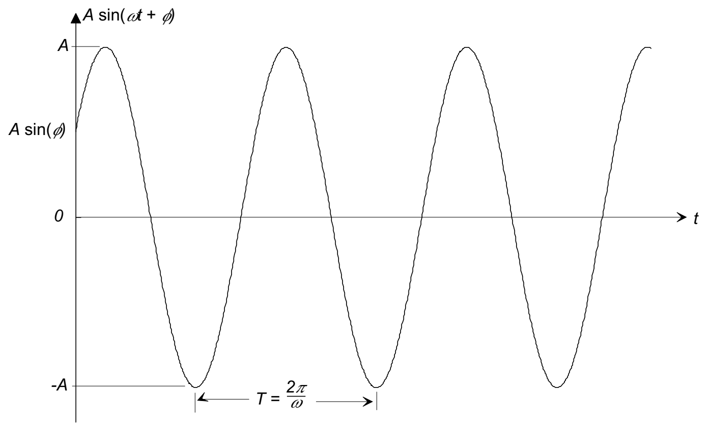
- Nise: 10.1

1 Sinusoidal Frequency Response

1.1 Definitions

Consider a sinusoidal waveform

$$f(t) = A \sin(\omega t + \phi)$$



where

A is the amplitude (in appropriate units)

ω is the angular frequency (rad/s)

ϕ is the phase (rad)

In addition we can define

T the period $T = 2\pi/\omega$ (s)

f the frequency, ($f = 1/T = \omega/2\pi$) (Hz)

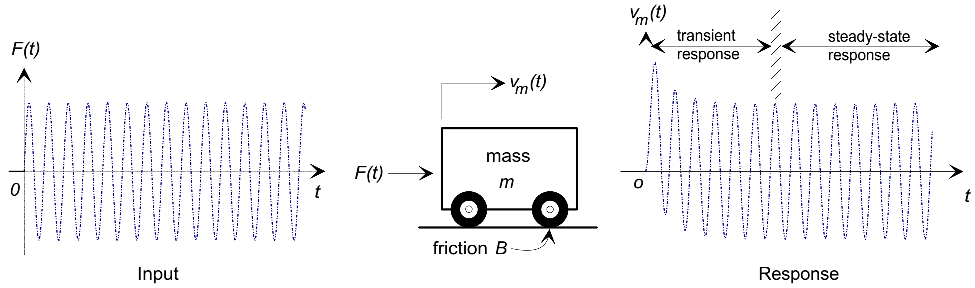
The Euler Formulas: We will frequently need the Euler formulas

$$\begin{aligned} e^{j\omega t} &= \cos(\omega t) + j\sin(\omega t) \\ e^{-j\omega t} &= \cos(\omega t) - j\sin(\omega t) \end{aligned}$$

or conversely

$$\begin{aligned} \cos(\omega t) &= \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) \\ \sin(\omega t) &= \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right) \end{aligned}$$

1.2 The Steady-State Sinusoidal Response



Assume a system, such as shown above, is excited by a sinusoidal input. The total response will have two components a) a transient component, and a steady-state component

$$y(t) = y_h(t) + y_p(t).$$

We define the steady-state component as the particular solution $y_p(t)$. Let the system differential equation be

$$a_n\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_1\frac{dy}{dt} + a_0 y = b_m\frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \ldots + b_1\frac{du}{dt} + b_0 u$$

with a complex exponential input

$$u(t) = e^{j\omega t}.$$

Assume a particular solution $y_p(t)$ to be of the same form as the input, that is

$$y_p(t) = Ae^{j\omega t}$$

and since

$$\frac{d^k y_p}{dt^k} = A(j\omega)^k e^{j\omega t}$$

substitution into the differential equation gives:

$$\begin{aligned} &\left(a_n(j\omega)^n + a_{n-1}(j\omega)^{n-1} + \ldots + a_1(j\omega) + a_0\right) Ae^{j\omega t} \\ &\qquad = \left(b_m(j\omega)^m + b_{m-1}(j\omega)^{m-1} + \ldots + (b_1 j\omega) + b_0\right) e^{j\omega t} \end{aligned}$$

or

$$A = \frac{b_m(j\omega)^m + b_{m-1}(j\omega)^{m-1} + \ldots + b_1(j\omega) + b_0}{a_n(j\omega)^n + a_{n-1}(j\omega)^{n-1} + \ldots + a_1(j\omega) + a_0}$$

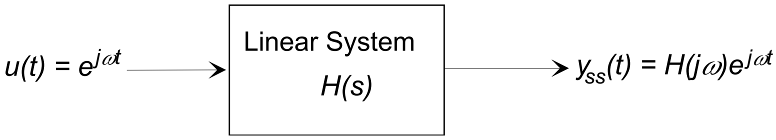
Examination of this equation shows its similarity to the transfer function $H(s)$, in fact

$$A = H(s)|_{s=j\omega} = H(j\omega)$$

so that the steady-state response $y_{ss}(t)$ is

$$\boxed{y_{ss}(t) = y_p(t) = Ae^{j\omega t} = H(j\omega)e^{j\omega t},} \tag{1}$$

or in other words, the steady-state response to a complex exponential input is defined by the *transfer function evaluated at* $s = j\omega$, or along the imaginary axis of the s-plane. Note that $H(j\omega)$ is in general complex.



We now extend this argument to a real sinusoidal input, for example $u(t) = \cos(\omega t) = (e^{j\omega t} + e^{-j\omega t})/2$. The principle of superposition for linear systems allows us to express the response as the sum of the two responses to the complex exponentials:

$$y_{ss}(t) = \frac{1}{2}\left(H(j\omega)e^{j\omega t} + H(-j\omega)e^{-j\omega t}\right)$$

We now proceed as follows:

- We show that $H(-j\omega) = \overline{H(j\omega)}$ where $\overline{H(j\omega)}$ denotes the complex conjugate (see the Appendix), so that

$$\boxed{y_{ss}(t) = \frac{1}{2}\left(H(j\omega)e^{j\omega t} + \overline{H(j\omega)}e^{-j\omega t}\right)} \tag{2}$$

- We break up $H(j\omega)$ into its real and imaginary parts,

$$\begin{aligned} H(j\omega) &= \Re\{H(j\omega)\} + j\Im\{H(j\omega)\} \\ \overline{H(j\omega)} &= \Re\{H(j\omega)\} - j\Im\{H(j\omega)\} \end{aligned}$$

  and use the Euler formula to write

$$\begin{aligned} e^{j\omega t} &= \cos(\omega t) + j\sin(\omega t) \\ e^{-j\omega t} &= \cos(\omega t) - j\sin(\omega t) \end{aligned}$$

- We combine the real and imaginary parts of Eq. (2) to conclude

$$\boxed{y_{ss}(t) = \Re\{H(j\omega)\}\cos(\omega t) - \Im\{H(j\omega)\}\sin(\omega t)} \tag{3}$$

- We then use the trig. identity

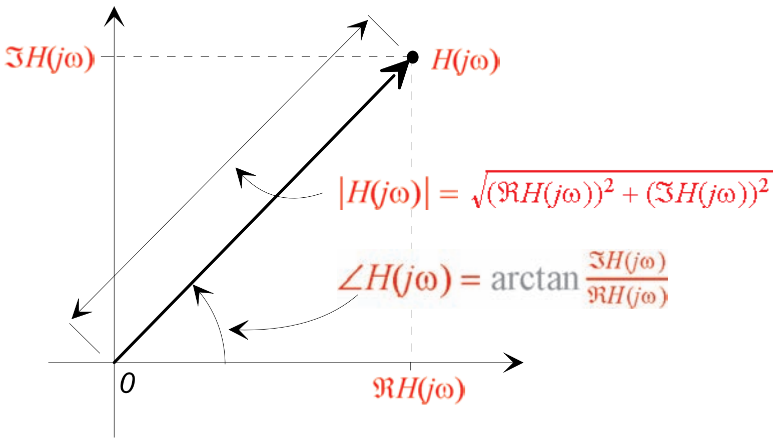
$$a\cos\theta - b\sin\theta = \sqrt{a^2+b^2}\cos(\theta+\phi)$$

to write Eq. (3) as

$$\boxed{y_{ss}(t) = |H(j\omega)|\cos\left(\omega t + \angle H(j\omega)\right)} \quad (4)$$

where

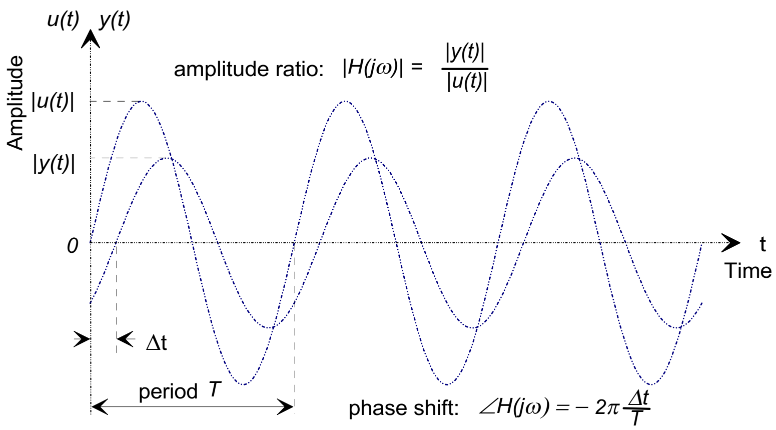
$$\begin{aligned} |H(j\omega)| &= \sqrt{\Re^2\{H(j\omega)\} + \Im^2\{H(j\omega)\}} \\ \angle H(j\omega) &= \arctan\left(\frac{\Im\{H(j\omega)\}}{\Re\{H(j\omega)\}}\right) \end{aligned}$$



Equation (4) states the answer we seek. It shows that

- The steady-state sinusoidal response is a sinusoid of the *same angular frequency* as the input,
- The response differs from the input by (i) a *change in amplitude* as defined by $|H(j\omega)|$, and (ii) an added *phase shift* $\angle H(j\omega)$.

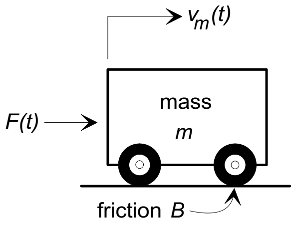
$H(j\omega)$ is known as the *frequency response function*. $|H(j\omega)|$ is the *magnitude* of the frequency response function, and $\angle H(j\omega)$ is the *phase*.



Note that if $|H(j\omega)| > 1$ the sinusoidal input is *amplified*, while if $|H(j\omega)| < 1$ the input is *attenuated* by the system.

■ Example 1

The mechanical system



has a transfer function

$$H(s) = \frac{v_m(s)}{F(s)} = \frac{1}{ms + B}$$

where $m = 1$ kg, and $B = 2$ Ns/m. Find the steady-state response if $F(t) = 10\sin(5t)$.

$$H(s) = \frac{1}{s+2}$$

so that the frequency response function is

$$H(j\omega) = H(s)|_{s=j\omega} = \frac{1}{j\omega + 2} = \frac{2 - j\omega}{\omega^2 + 4}$$

Then

$$|H(j\omega)| = \frac{1}{\sqrt{\omega^2+4}}, \qquad \angle H(j\omega) = \arctan\left(-\frac{\omega}{2}\right).$$

With $\omega = 5$ rad/s,

$$\begin{aligned} v_{ss}(t) &= 10\,|H(j\omega)|\sin(5t + \angle H(j\omega) \\ &= \frac{10}{\sqrt{29}}\sin(5t - \arctan 2.5) \\ &= 1.857\sin(5t - 1.1903) \end{aligned}$$

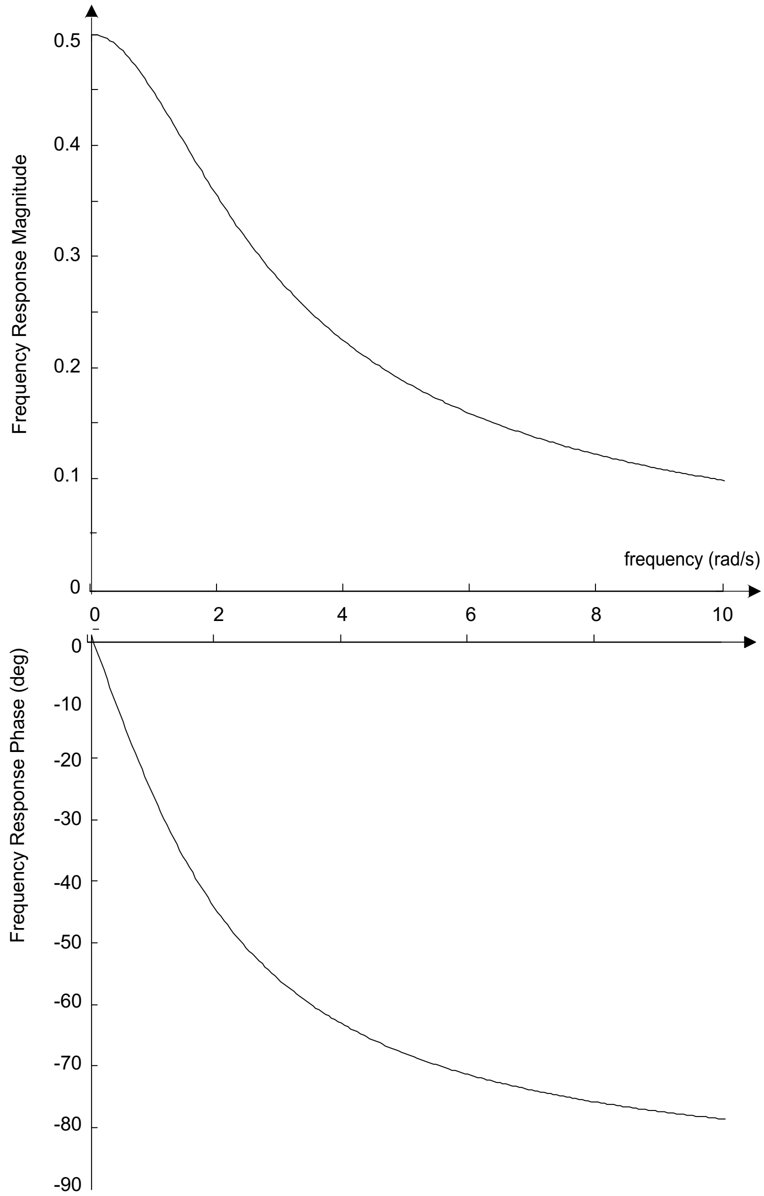
■ Example 2

Plot the variation of $|H(j\omega)|$ and $\angle H(j\omega)$ from $\omega = 0$ to 10 rad/s.

From above

$$|H(j\omega)| = \frac{1}{\sqrt{\omega^2+4}}, \quad \text{and} \quad \angle H(j\omega) = \arctan\left(-\frac{\omega}{2}\right).$$

These functions are plotted below:



Note that

- As the input frequency ω increases, the response *magnitude* decreases.
- At low frequencies the *phase* is a small negative number, but as the frequency increases the phase lag increases and apparently is tending toward $-90°$ at high frequencies.

Appendix: Evaluation of $H(-j\omega)$.

We start with

$$H(j\omega) = \frac{b_m(j\omega)^m + b_{m-1}(j\omega)^{m-1} + \ldots + b_1(j\omega) + b_0}{a_n(j\omega)^n + a_{n-1}(j\omega)^{n-1} + \ldots + a_1(j\omega) + a_0}$$

so that

$$H(-j\omega) = \frac{b_m(-j\omega)^m + b_{m-1}(-j\omega)^{m-1} + \ldots + b_1(-j\omega) + b_0}{a_n(-j\omega)^n + a_{n-1}(-j\omega)^{n-1} + \ldots + a_1(-j\omega) + a_0}$$

Note that

$$\begin{array}{lcll} (j\omega)^k & = & (-1)^{k/2}\omega^k & k \text{ even} \\ & & j(-1)^{(k-1)/2}\omega^k & k \text{ odd} \\ & & & \\ (-j\omega)^k & = & (-1)^{k/2}\omega^k & k \text{ even} \\ & & -j(-1)^{(k-1)/2}\omega^k & k \text{ odd} \end{array}$$

Thus in both $H(j\omega)$ and $H(-j\omega)$

- The terms with even powers of $\pm j\omega$ in the numerator and denominator of $H(j\omega)$ and $H(-j\omega)$ generate real terms, while
- the terms with odd powers of $\pm j\omega$ generate imaginary terms.

With these substitutions, comparison of $H(j\omega)$ and $H(-j\omega)$ shows

- The real terms (even powers of $\pm j\omega$) are the same, while
- The imaginary terms (odd powers of $\pm j\omega$) have opposite signs

leading to the conclusion

$$\boxed{H(-j\omega) = \overline{H(j\omega)}.}$$