2.016 Hydrodynamics

PS #1 Fall 2005

1. a) Temperature <u>is</u> isotropic, because it is the same in all directions. It is a scalar quantity.

b) Pressure <u>is</u> isotropic.

c) Shear stress is <u>not</u> isotropic, because it is a vector. Shear stress has directionality.

d) Dynamic viscosity of common fluids like water, air, and oil <u>is</u> isotropic. These fluids are called Newtonian, because they obey the relation

$$\tau = \mu \frac{du}{dy}$$

dynamic viscosity, scalar, same in all directions.

Non-Newtonian fluids have more interesting behavior, and you can learn more in 2.341J Macromolecular Hydrodynamics.

<u>Note</u>: isotropic = same in all directions

isentropic = at constant entropy.

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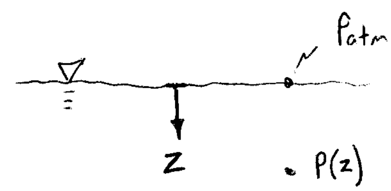
2. $\rho = \rho_0 + m z^2$

$$\frac{dp}{dz} = +\rho g$$

since we define z as positive downwards, then pressure increases as z increases.



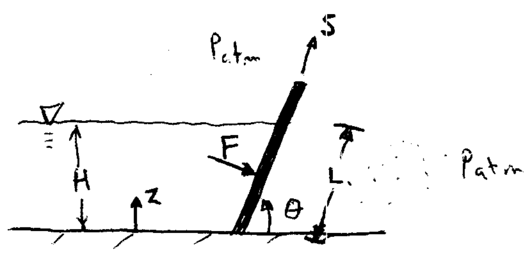
$$\int_{P_{atm}}^{P} dp = g\int_0^z \rho\, dz = g\int_0^z (\rho_0 + m z^2)\, dz$$

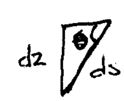
$$P - P_{atm} = g\left[\rho_0 z + \tfrac{1}{3} m z^3\right]_0^z$$

$$\boxed{P = P_{atm} + \rho_0 g z + \tfrac{1}{3} m g z^3}$$

Note: $z > 0$ under the surface, so P increases as you go down.

3.





$dz = ds \cdot \sin\theta$

- Since atmospheric pressure acts on the backside of the wall, use gauge pressure to find the force.
- Pressure acts normal to the surface.
- Take $z = 0$ at the bottom to simplify the math.

$$F = \int_0^L dF = \int_0^L P w\, ds = \int_0^H P w \frac{dz}{\sin\theta}$$

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3 cont.

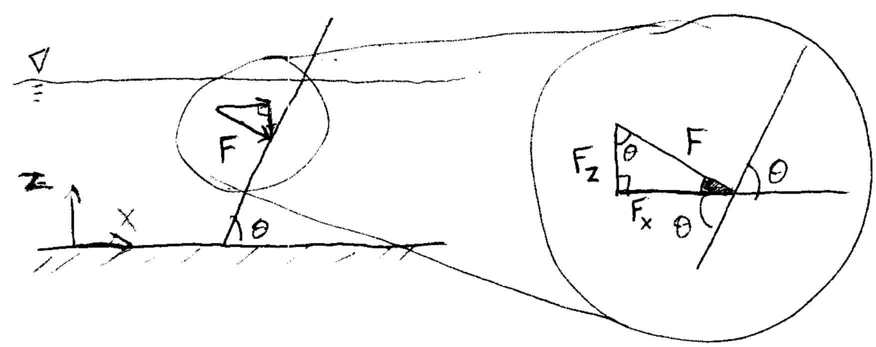
$$F = \int_0^H \frac{P\,w}{\sin\theta}\,dz$$

Use gauge pressure: $P = \rho g (H - z)$

$$F = \int_0^H \frac{\rho g w}{\sin\theta}(H - z)\,dz$$

$$= \frac{\rho g w}{\sin\theta}\left(Hz - \frac{1}{2}z^2\right)\Big|_0^H$$

$$\boxed{F = \frac{1}{2}\frac{\rho g w H^2}{\sin\theta}}$$ total resulting force



$F_z = -F\cos\theta$

$F_x = F\sin\theta$

$$\boxed{F_z} = F\cos\theta = \boxed{-\frac{1}{2}\rho g w H^2 \cot\theta}$$ vertical force

$$\boxed{F_x = \frac{1}{2}\rho g w H^2}$$ horizontal force

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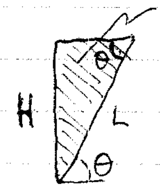
3 cont. Could alternatively consider weight of water on wall:



$$A = \tfrac{1}{2}(L\cos\theta)\,H = \tfrac{1}{2}\cot\theta\,H^2 \qquad L = \frac{h}{\sin\theta}$$

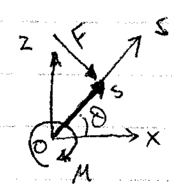
$$m = \rho \forall = \rho A \cdot w = \tfrac{1}{2}\rho w \cot\theta\, H^2$$

$$F_z = mg = \tfrac{1}{2}\rho g w \cot\theta\, H^2 \quad \checkmark$$

b) To find center of pressure, determine location where the resulting moment is zero. This will be the location of the average moment, so to speak (prove it for yourself, or wave hands...)

$$\vec{M} = \vec{s} \times \vec{F}$$

$$S_{cp} = \frac{M_o}{F}$$



$z = s\cdot\sin\theta$

$h = L\sin\theta$

$F = \frac{1}{2}\frac{\rho g w H^2}{\sin\theta} = \frac{1}{2}\rho g w L^2 \sin\theta$

To find M: $$M_o = \int_0^L \vec{s} \times d\vec{F} = \int_0^L s \cdot P w\, ds = \int_0^L s \cdot \rho g w (h - z)\, ds$$

$$= \rho g w \int_0^L (h s - s^2 \cdot \sin\theta)\, ds$$

$$= \rho g w \left(\tfrac{1}{2}\underbrace{h}_{L\sin\theta} s^2 - \tfrac{1}{3}s^3\sin\theta\right)\Big|_0^L = \rho g w\left(\tfrac{1}{2}L^3\sin\theta - \tfrac{1}{3}L^3\sin\theta\right)$$

$$M_o = \tfrac{1}{6}\rho g w L^3 \sin\theta$$

$$\boxed{S_{cp}} = \frac{M_o}{F} = \frac{\frac{1}{6}\rho g w L^3\sin\theta}{\frac{1}{2}\rho g w L^2 \sin\theta} = \boxed{\tfrac{1}{3}L}$$

Note: This is the same as for a vertical wall. Groovy.

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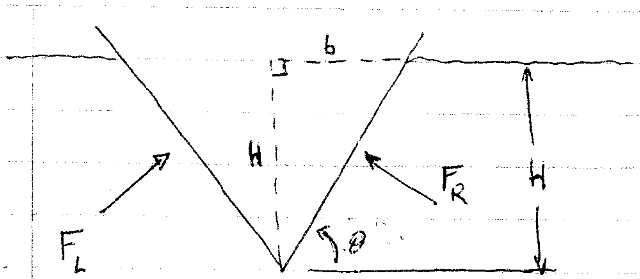
4. Let's prove Αρχιμηδης' principle – the bouyancy force on a submerged body equals the weight of the displaced fluid

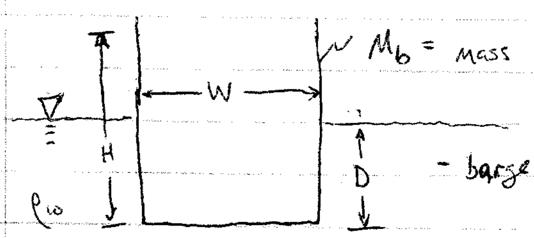


- static equilibrium
- symmetry left & right.

$$W = mg = \rho \forall g = \rho\ 2\cdot\left(\tfrac{1}{2} bH\cdot w\right) g = \rho g w H^2 \cot\theta$$

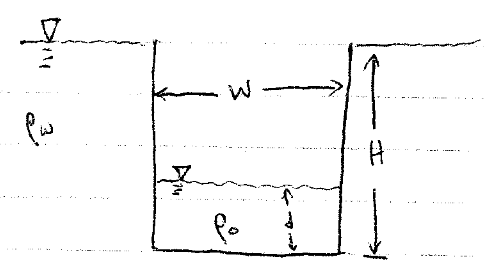
$$F_z = F_{z_L} + F_{z_R} = \underbrace{\tfrac{1}{2}\rho g w H^2 \cot\theta + \tfrac{1}{2}\rho g w H^2 \cot\theta}_{\text{From problem 3.}} = \rho g w H^2 \cot\theta \quad ✓$$

5.



M_b = mass of empty barge

- barge extends L into page



By Archimedes' principle, $M_b g = M_w g = \rho_w W D L g$

For sinking barge,

$$M_b g + \rho_o W d L g = \rho_w W H L g$$

$$\rho_w \cancel{W} D \cancel{L} \cancel{g} + \rho_o \cancel{W} d \cancel{L} \cancel{g} = \rho_w \cancel{W} H \cancel{L} \cancel{g}$$

$$d = \frac{\rho_w H - \rho_w D}{\rho_o}$$

$$\boxed{d = \frac{\rho_w}{\rho_o}(H - D)}$$

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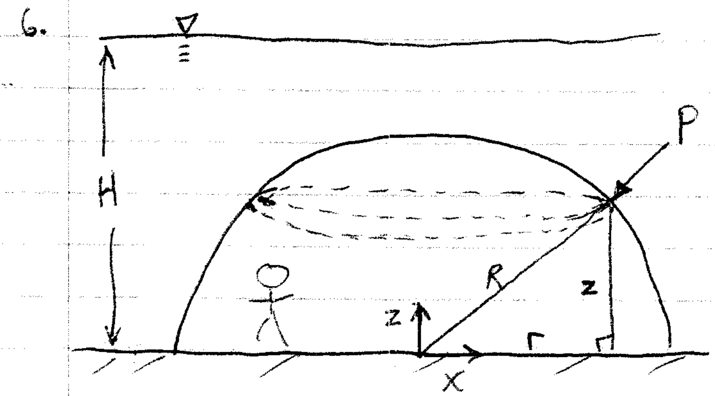
6.



- assume pressure is axisymmetric
- use gauge pressure with $z=0$ at sea floor

a) $\boxed{P(z) = \rho g (H - z)}$ pressure acts normal to the surface.

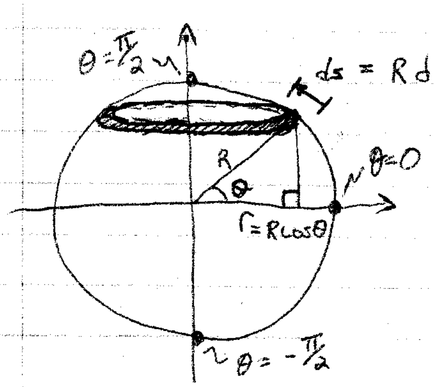
or

$\boxed{P(\theta) = \rho g (H - R\sin\theta)}$ pressure still acts normal to the surface.

b) To find the force, integrate the pressure over the surface area... and be clever:

$\boxed{F_x = 0}$ by symmetry

To find the force in the z-direction, integrate the pressure over the surface area projected into the z-direction. Before we do that, let's review how to find the surface area of a sphere.



$$A = \int 2\pi r\,ds = \int_{-\pi/2}^{\pi/2} 2\pi (R\cos\theta)\cdot(R\,d\theta)$$

$$= 2\pi R^2 \int_{-\pi/2}^{\pi/2} \cos\theta\,d\theta = 2\pi R^2 \sin\theta \Big|_{-\pi/2}^{\pi/2}$$

$$A = 4\pi R^2 \checkmark$$

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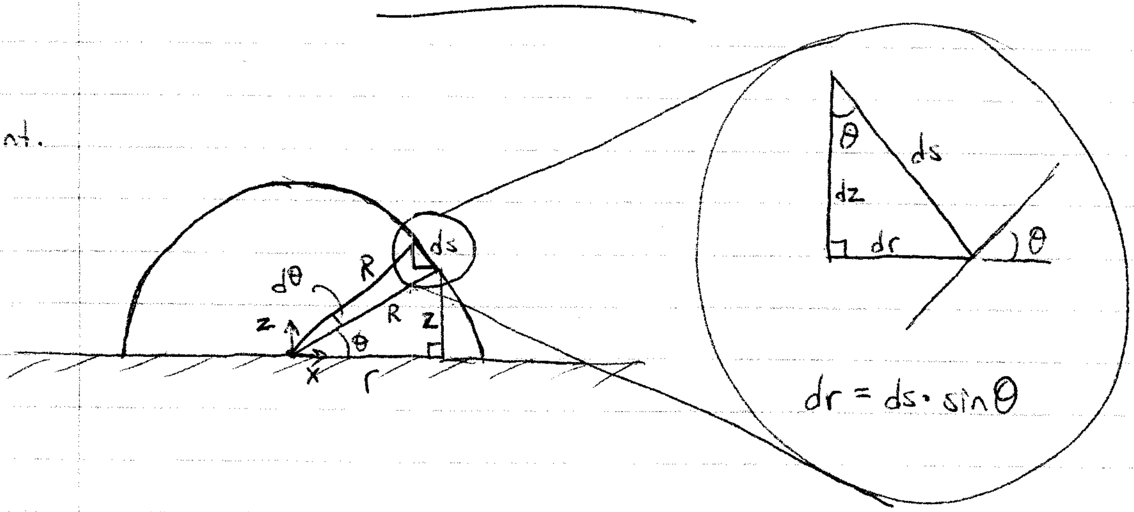
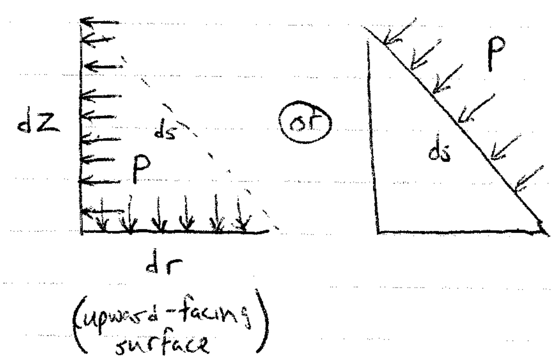
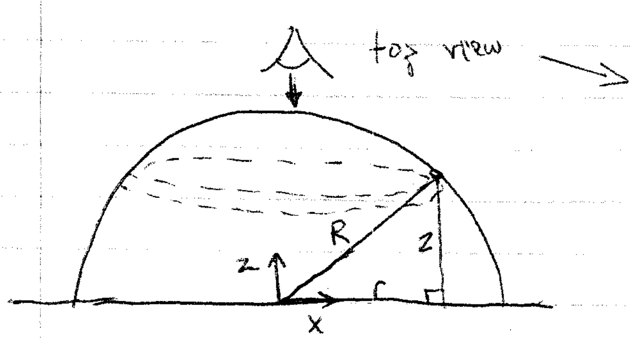
6 cont.

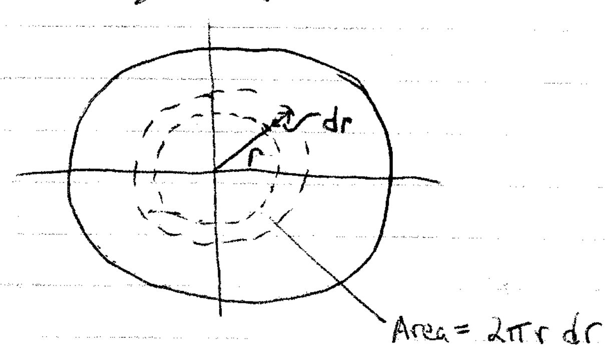


At any point on the surface of the sphere, you can think of the pressure acting on an imaginary upward-facing surface and an imaginary sideways-facing surface. To find the force in the z-direction integrate the pressure over the upward-facing (i.e. z-direction) surface!





top view



$$F_z = \int_A P\, dA = \int P\, 2\pi r\, dr = \int P\, 2\pi\, (R\cos\theta)\cdot(\sin\theta\, ds)$$

$$= \int P\, 2\pi\, (R\cos\theta)\cdot(\sin\theta\, R\, d\theta)$$

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6 cont.

$$F_z = \int_0^{\pi/2} P(\theta)\, 2\pi R^2 \sin\theta \cos\theta \, d\theta$$

$$= -\int_0^{\pi/2} \pi R^2 \rho g \left(2H\sin\theta\cos\theta - 2R\sin^2\theta\cos\theta\right) d\theta$$

negative sign because force is pushing down.

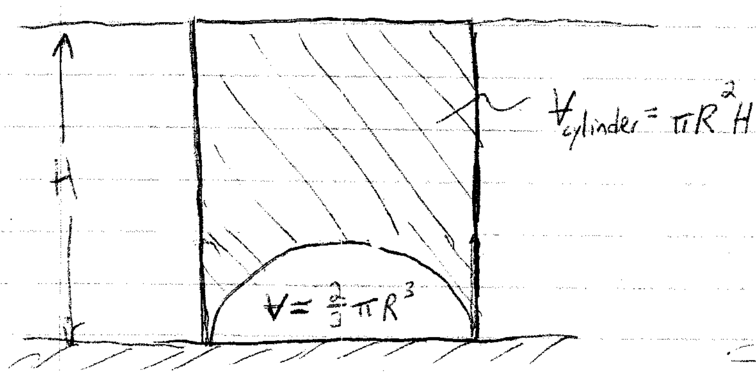
$$= -\pi R^2 \rho g \int_0^{\pi/2} \left(H\sin(2\theta) - 2R\sin^2\theta\cos\theta\right) d\theta$$

$$= -\pi R^2 \rho g \left[-\tfrac{1}{2}H\cos(2\theta) - \tfrac{2}{3}R\sin^3\theta\right]_0^{\pi/2}$$

$$= -\pi R^2 \rho g \left[(-\tfrac{1}{2})H(-1) - \tfrac{2}{3}R(1)^3 + \tfrac{1}{2}H(1) + 0\right]$$

$$\boxed{F_z = -\pi R^2 \rho g \left(H - \tfrac{2}{3}R\right)}$$

Another, more intuitive approach is that F_z = weight of water on top of dome.



$= -\rho g \cdot$(volume of water on top)

$= -\rho g\,(\forall_{cylinder} - \forall_{hemisphere})$

$= -\rho g\,(\pi R^2 H - \tfrac{2}{3}\pi R^3)$

$$\boxed{F_z = -\pi R^2 \rho g \left(H - \tfrac{2}{3}R\right)}$$