2.016 Hydrodynamics

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0.1 Derivation of Added Mass around a sphere

When a body moves in a fluid, some amount of fluid must move around it. When the body accelerates, so too must the fluid. Thus, more force is required to accelerate the body in the fluid than in a vacuum. Since force equals mass times acceleration, we can think of the additional force in terms of an imaginary *added mass* of the object in the fluid.

One can derive the added mass of an object by considering the hydrodynamic force acting on it as it accelerates. Consider a sphere of radius, R, accelerating at rate $a = \partial U/\partial t = \dot{U}$. We find the hydrodynamic force in the x-direction by integrating the pressure over the area projected in the x-direction:

$$\vec{F_x} = \int P d\vec{A}_x$$

where

- $d\vec{A}_x = \cos\theta dA$

  $dA = 2\pi r ds$

  $r = R\sin\theta$

  $ds = Rd\theta$

- $P = -\rho\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}|\vec{\nabla}\phi|^2\right]$, by unsteady Bernoulli's equation

  $\phi = U\cos\theta\frac{R^3}{2r^2}$, for axisymmetric flow around a sphere

  $\frac{\partial\phi}{\partial t}|_{r=R} = \dot{U}\cos\theta\frac{R^3}{2r^2} = \dot{U}\cos\theta\frac{R}{2}$

  $\frac{1}{2}|\vec{\nabla}\phi|^2|_{r=R} = \frac{1}{2}|(-U\cos\theta\frac{R^3}{r^3}, -U\sin\theta\frac{R^3}{2r^3})|^2 = \frac{1}{2}[U^2\cos^2\theta + \frac{1}{4}U^2\sin^2\theta]$

so

$$\begin{aligned}
F_x &= \int_0^\pi \left[-\rho\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}|\vec{\nabla}\phi|^2\right]\right]\cos\theta 2\pi R^2\sin\theta d\theta \\
&= \int_0^\pi \left[-\rho\left[\dot{U}\cos\theta\frac{R}{2} + \frac{1}{2}(U^2\cos^2\theta + \frac{1}{4}U^2\sin^2\theta)\right]\right]\cos\theta 2\pi R^2\sin\theta d\theta \\
&= -\rho\cdot 2\pi R^2\cdot\dot{U}\frac{R}{2}\underbrace{\int_0^\pi \sin\theta\cos^2\theta d\theta}_{=2/3} - \rho\cdot 2\pi R^2\cdot\frac{1}{2}U^2\underbrace{\int_0^\pi [\sin\theta\cos^3\theta + \frac{1}{4}\sin^3\theta\cos\theta]d\theta}_{=0} \\
&= -\frac{2}{3}\rho\pi R^3\dot{U}
\end{aligned}$$

where $a = \dot{U}$ is the acceleration of the body, and the negative sign indicates that the force is in the negative x-direction, opposing the acceleration. Thus, the body must exert this extra force, and the apparent *added mass* is

$$m_a = \frac{2}{3}\rho\pi R^3$$

Similarly, for a cylinder of radius, R, and length, L, accelerating at rate $a = \partial U/\partial t = \dot{U}$. We find the hydrodynamic force in the x-direction by integrating the pressure over the area projected in the x-direction:

$$\vec{F_x} = \int P d\vec{A}_x$$

where

- $d\vec{A}_x = \cos\theta dA$

  $dA = Lds$

  $ds = Rd\theta$

- $P = -\rho\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}|\vec{\nabla}\phi|^2\right]$, by unsteady Bernoulli's equation

  $\phi = U\frac{R^2}{r}\cos\theta$, for flow around a cylinder

  $\frac{\partial\phi}{\partial t}|_{r=R} = \dot{U}\frac{R^2}{r}\cos\theta = \dot{U}R\cos\theta$

  $\frac{1}{2}|\vec{\nabla}\phi|^2|_{r=R} = \frac{1}{2}|(-U\frac{R^2}{r^2}\cos\theta, -U\frac{R^2}{r^2}\sin\theta)|^2 = \frac{1}{2}U^2$

so

$$\begin{aligned} F_x &= \int_0^{2\pi}\left[-\rho\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}|\vec{\nabla}\phi|^2\right]\right]\cos\theta RLd\theta \\ &= \int_0^{2\pi}\left[-\rho\left[\dot{U}R\cos\theta + \frac{1}{2}U^2)\right]\right]\cos\theta RLd\theta \\ &= -\rho\cdot RL\cdot\dot{U}R\underbrace{\int_0^{2\pi}\cos^2\theta d\theta}_{=\pi} - \rho\cdot RL\cdot\frac{1}{2}U^2\underbrace{\int_0^{2\pi}\cos\theta d\theta}_{=0} \\ &= -\rho\pi R^2 L\dot{U} \end{aligned}$$

where $a = \dot{U}$ is the acceleration of the body, and the negative sign indicates that the force is in the negative x-direction, opposing the acceleration. Thus, the body must exert this extra force, and the apparent *added mass* is

$$m_a = \rho\pi R^2 L$$