2.016 Hydrodynamics
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0.1 Derivation of unsteady Bernoulli's Equation

Conservation of Momentum says

$$m\vec{a} = \vec{F}$$

so

$$\rho\vec{a} = \rho\frac{D\vec{V}}{Dt} = \frac{\vec{F}}{\forall}$$

This is the acceleration and forces acting on Bob the Fluid Blob. The total derivative of the velocity is expanded like this:

$$\frac{D\vec{V}(t,x,y,z)}{Dt} = \frac{\partial\vec{V}}{\partial t} + \frac{\partial\vec{V}}{\partial x}\underbrace{\frac{\partial x}{\partial t}}_{u} + \frac{\partial\vec{V}}{\partial y}\underbrace{\frac{\partial y}{\partial t}}_{v} + \frac{\partial\vec{V}}{\partial z}\underbrace{\frac{\partial z}{\partial t}}_{w}$$

$$\begin{aligned}
\frac{D\vec{V}}{Dt} &= \frac{\partial\vec{V}}{\partial t} + u\frac{\partial\vec{V}}{\partial x} + v\frac{\partial\vec{V}}{\partial y} + w\frac{\partial\vec{V}}{\partial z} \\
&= \frac{\partial\vec{V}}{\partial t} + \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)\vec{V} \\
&= \frac{\partial\vec{V}}{\partial t} + \left((u,v,w)\cdot\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\right)\vec{V} \\
\frac{D\vec{V}}{Dt} &= \frac{\partial\vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V}
\end{aligned}$$

For irrotational flow, $(\vec{\nabla}\times\vec{V} = 0)$, so $(\vec{V}\cdot\vec{\nabla})\vec{V} = \vec{\nabla}(\frac{1}{2}\vec{V}\cdot\vec{V})$ and

$$\frac{D\vec{V}}{Dt} = \frac{\partial\vec{V}}{\partial t} + \vec{\nabla}\left(\frac{1}{2}\vec{V}\cdot\vec{V}\right)$$

Also for irrotational flow, we can use the velocity potential $\vec{V} = \vec{\nabla}\phi$, and we have

$$\rho\frac{D\vec{V}}{Dt} = \rho\left[\frac{\partial\vec{\nabla}\phi}{\partial t} + \vec{\nabla}\left(\frac{1}{2}\vec{\nabla}\phi\cdot\vec{\nabla}\phi\right)\right]$$

The forces acting on Bob are pressure and gravity, so the momentum equation becomes

$$\rho\left[\frac{\partial\vec{\nabla}\phi}{\partial t} + \vec{\nabla}\left(\frac{1}{2}\vec{\nabla}\phi\cdot\vec{\nabla}\phi\right)\right] = -\vec{\nabla}p - \underbrace{\rho g}_{\frac{d}{dz}(\rho g z)}\hat{k} = -\vec{\nabla}p - \vec{\nabla}(\rho g z)$$

$$\vec{\nabla}\left[\rho\frac{\partial\phi}{\partial t} + \frac{1}{2}\rho\left(\vec{\nabla}\phi\cdot\vec{\nabla}\phi\right) + p + \rho g z\right] = 0$$

And in one last glorious step, we integrate all the spacial derivatives (i.e. knock the nabla out), and we have the unsteady Bernoulli's Equation;

$$\rho\frac{\partial\phi}{\partial t} + \frac{1}{2}\rho\left(\vec{\nabla}\phi\cdot\vec{\nabla}\phi\right) + p + \rho g z = F(t)$$

where $F(t)$ is some function of t (is the "constant of integration").