Goals for today

- Feedback topology
  - Negative *vs* positive feedback

- Example of a system with feedback
  - Derivation of the closed-loop transfer function
  - Specification of the transient response by selecting the feedback gain

- Steady-state error
  - General steady error determination
  - Steady-state error with external disturbance input

- MATLABÈ LTI system analysis

Why feedback?

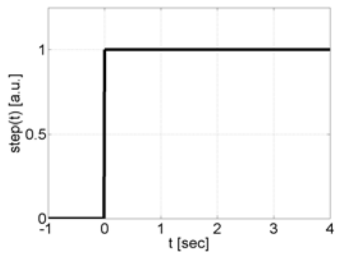
- Two reasons:
  - to make the system output resemble as much as possible a given input (“tracking”)
    - example: target-tracking missiles
  - to reduce the effect of disturbances in the system output
    - example: minute adjustments in steering wheel while we drive

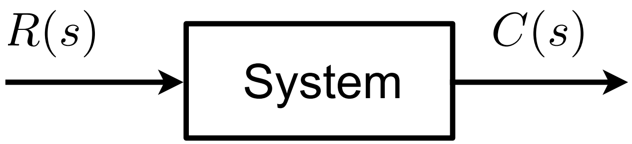
- Today: we examine the first case only (tracking)

- We will discuss disturbances in a future lecture

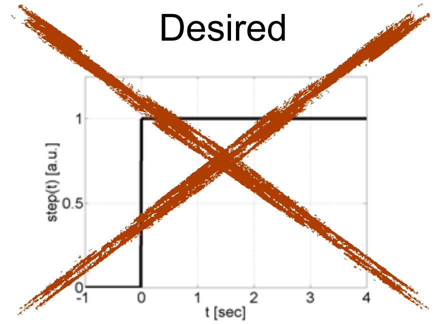
The problem with system dynamics

(for example)



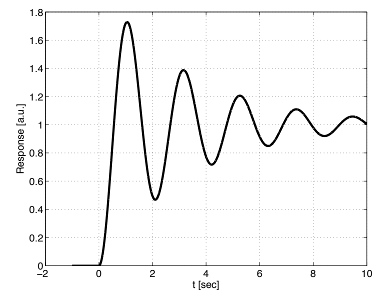


Desired



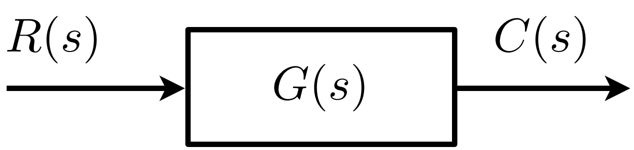
Actual

$C(s)$



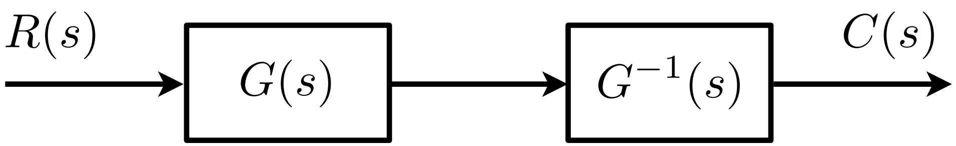
Why?

Because system response is determined by system transfer function G(s)



$$C(s) = G(s)R(s) \neq R(s)$$

~~Suggested solution: invert the Transfer Function~~



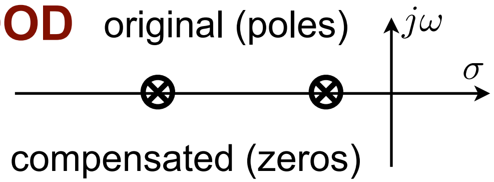
Now indeed

$$C(s) = G(s)\,G^{-1}(s)R(s) = R(s)$$

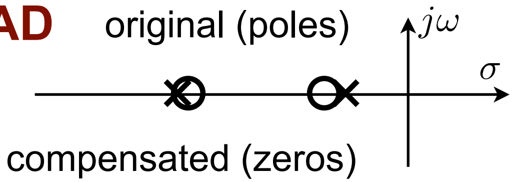
Problems

➡ What if we don't know the transfer function exactly?
(or if there is some variation among systems manufactured in a factory)

GOOD



BAD



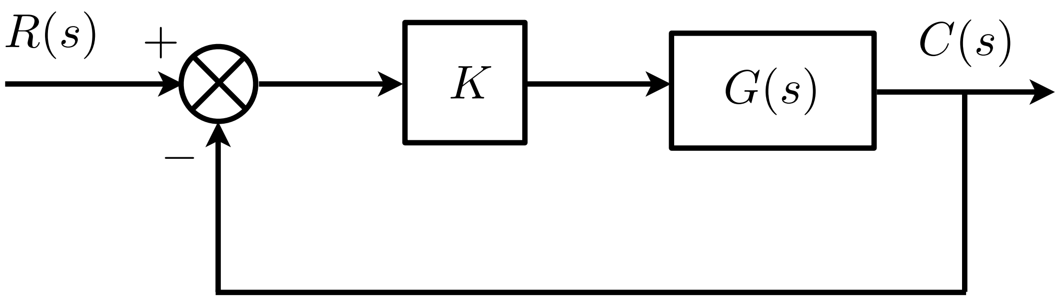
➡ Systems with more zeros than poles are not physically realizable
(but they are realizable if we use digital implementation)

E.g. $G(s) = \dfrac{s}{(s+p_1)(s+p_2)}$ — more poles

$G^{-1}(s) = \dfrac{(s+p_1)(s+p_2)}{s}$ — more zeros

➡ Disturbances are amplified (we will talk about these later)

Instead: feedback (sort-of) “inverts” the TF



K : Feedback “gain”

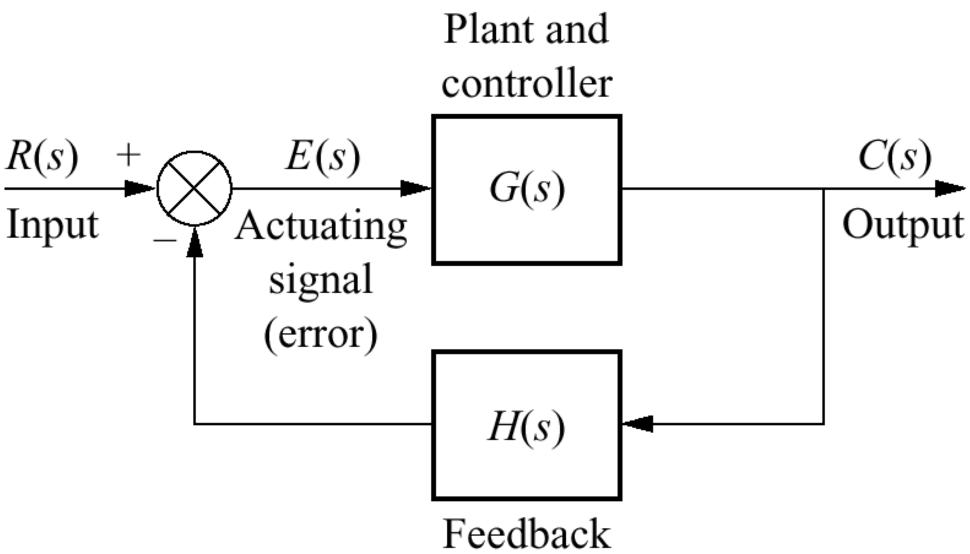
$G(s)$: “Open Loop (OL)” transfer function

“Closed Loop” transfer function $\dfrac{C(s)}{R(s)} = ?$

$$C(s) = KG(s)\Big(R(s) - C(s)\Big) \Rightarrow \boxed{\frac{C(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)}}$$

Note: if $K \to \infty \Rightarrow C(s) \approx R(s) \Rightarrow$ Tracking!!

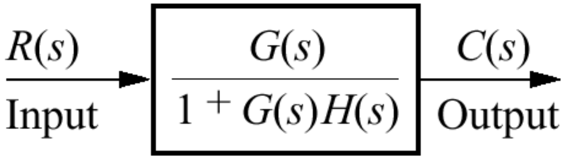
Negative feedback loop with feedback controller



(b)

$$
\begin{aligned}
E(s) &= R(s) - H(s)C(s) \\
C(s) &= E(s)G(s) \\
\Rightarrow C(s) &= \left[R(s) - H(s)C(s)\right] G(s) \\
\Rightarrow \frac{C(s)}{R(s)} &= \frac{G(s)}{1 + G(s)H(s)}. \\
\frac{C(s)}{R(s)} &: \text{Closed–loop TF} \\
G(s)H(s) &: \text{Open–loop TF.}
\end{aligned}
$$

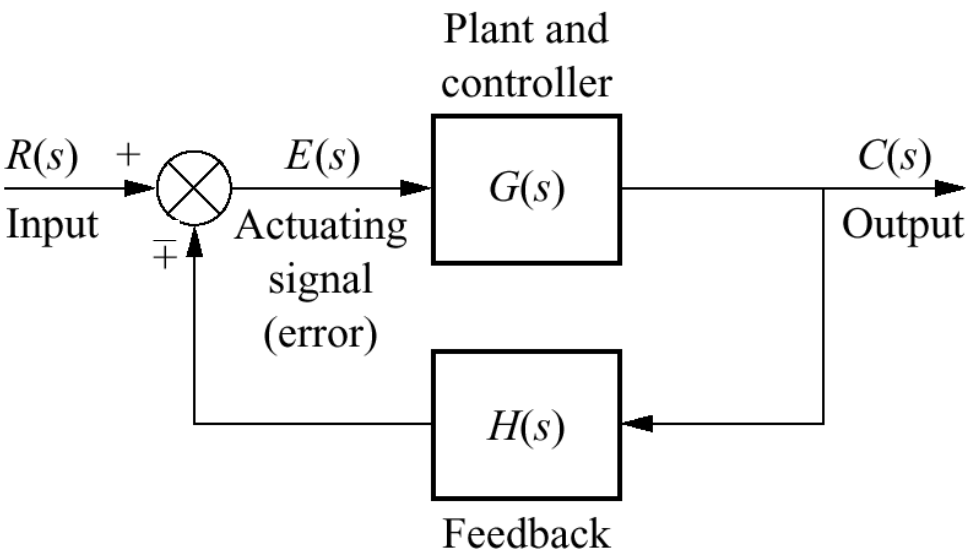
Equivalent system:



(c)

Nise Figure 5.6

Positive feedback



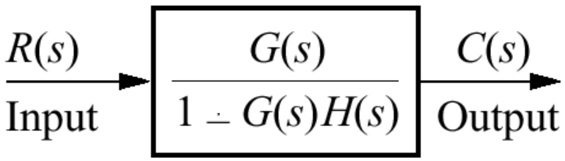
(b)

$$
\begin{aligned}
E(s) &= R(s) + H(s)C(s) \\
C(s) &= E(s)G(s) \\
\Rightarrow C(s) &= \left[R(s) + H(s)C(s)\right] G(s) \\
\Rightarrow \frac{C(s)}{R(s)} &= \frac{G(s)}{1 - G(s)H(s)}.
\end{aligned}
$$

$\frac{C(s)}{R(s)}$: Closed–loop TF

$G(s)H(s)$: Open–loop TF.

Equivalent system:

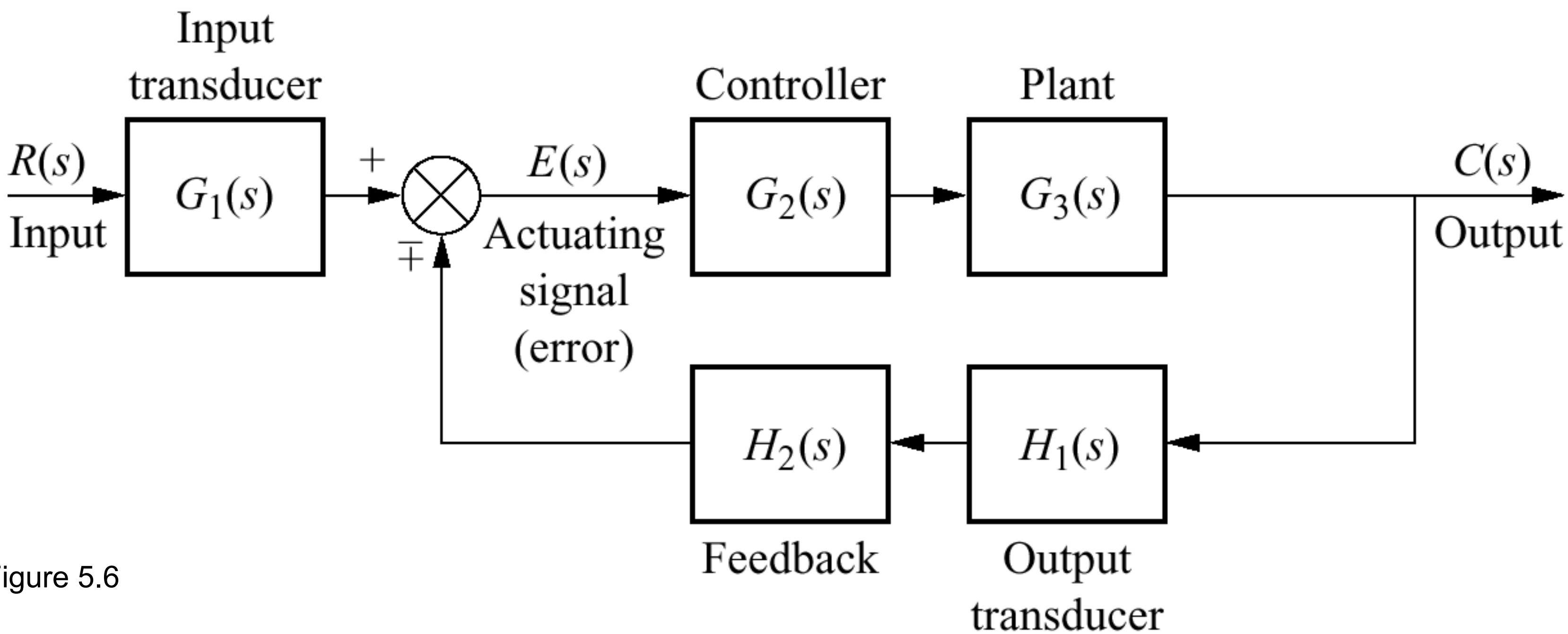


Nise Figure 5.6

(c)

Generally, positive feedback is dangerous: it may lead to unstable response (i.e. exponentially increasing) if not used with care

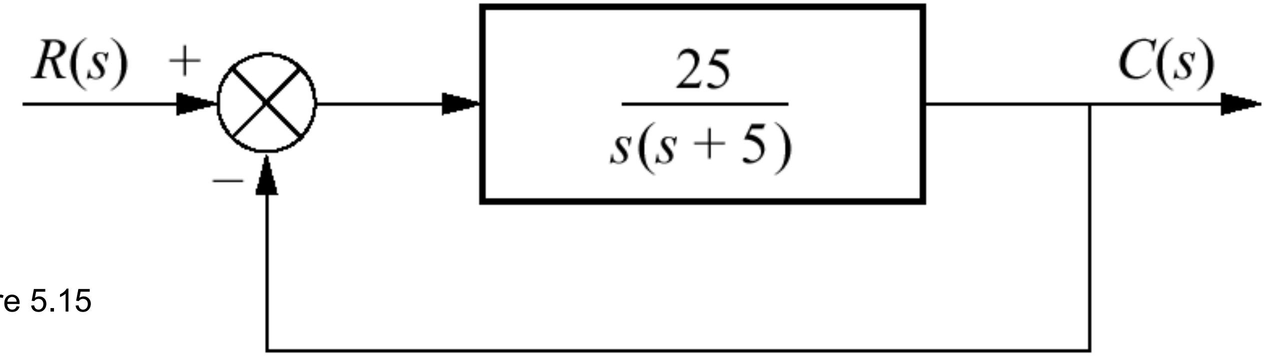
A more general feedback system



Nise Figure 5.6

Plant: the system we want to control (e.g., elevator plant: input=voltage, output=elevator position)
Controller: apparatus that produces input to plant (i.e. voltage to elevator's motor)
Transducers: converting physical quantities so the system can use them
(e.g., input transducer: floor button pushed→voltage;
output transducer: current elevator position →voltage)
Feedback: apparatus that contributes current system state to error signal (e.g., in elevator system, error=voltage representing desired position – voltage representing current position

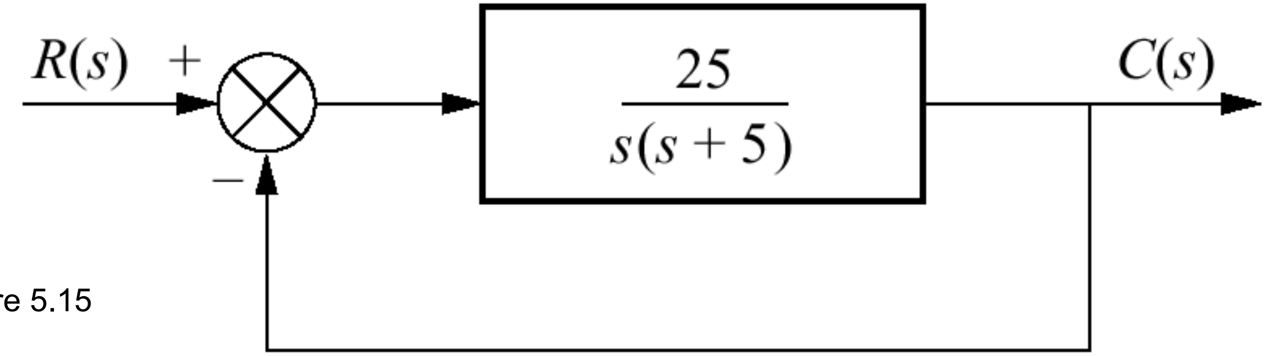
Transient response of a feedback system



Nise Figure 5.15

$$
\begin{aligned}
\text{Plant \& Controller} \quad G(s) &= \frac{25}{s(s+5)}, \\
\text{Gain} \quad K &= 25, \\
\text{Feedback} \quad H(s) &= 1. \\
\text{Open loop TF} \quad G(s)H(s) &= \frac{25}{s(s+5)}; \\
\text{Closed loop TF} \quad \frac{G(s)}{1+G(s)H(s)} &= \frac{25}{s^2+5s+25}.
\end{aligned}
$$

Transient response of a feedback system



Nise Figure 5.15

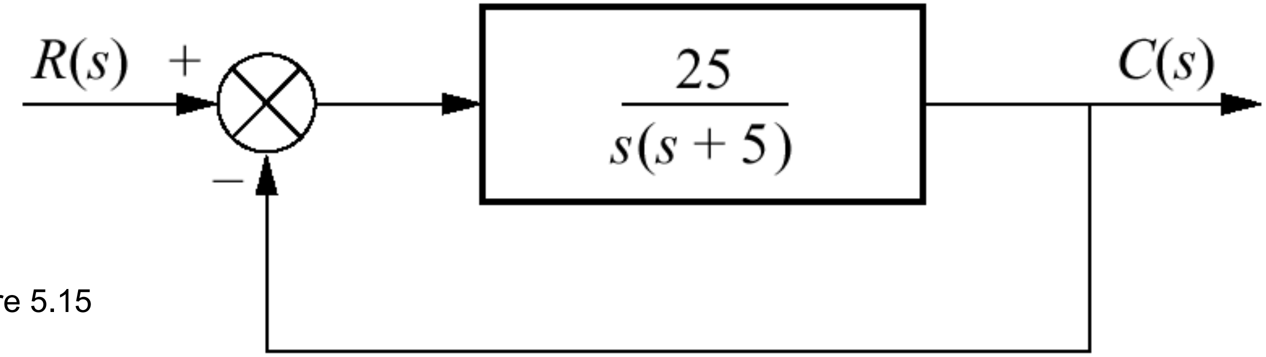
$$T(s) = \frac{G(s)}{1+G(s)H(s)} = \frac{25}{s^2+5s+25}.$$

$$\text{Recall} \quad \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$\Rightarrow \omega_n = \sqrt{25} = 5 \quad \text{rad/sec}; \qquad 2\zeta\omega_n = 5 \Rightarrow \zeta = 0.5;$$

⇒ The feedback system is **underdamped.**

Transient response of a feedback system



Nise Figure 5.15

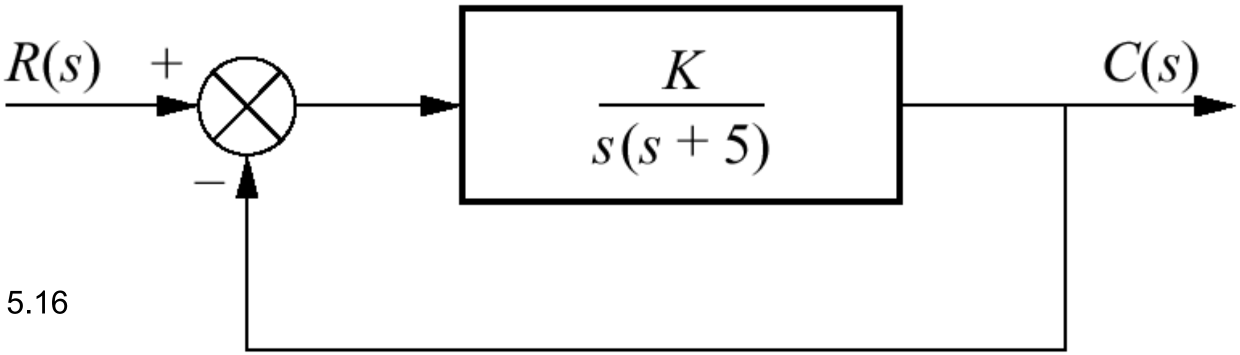
$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{25}{s^2 + 5s + 25}.$$

$$\text{Peak time} \quad T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.726 \quad \text{sec},$$

$$\%\text{OS} = \exp\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right) \times 100 = 16.3,$$

$$\text{Settling time} \quad T_s = \frac{4}{\zeta\omega_n} = 1.6 \quad \text{sec.}$$

Adjusting the transient by feedback



Nise Figure 5.16

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{K}{s^2 + 5s + K}.$$

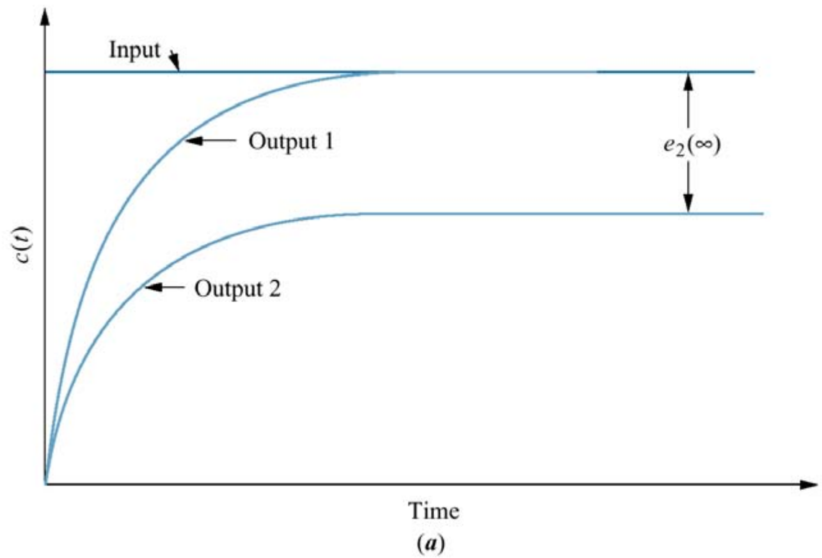
We wish to reduce overshoot to %OS=10% or less.

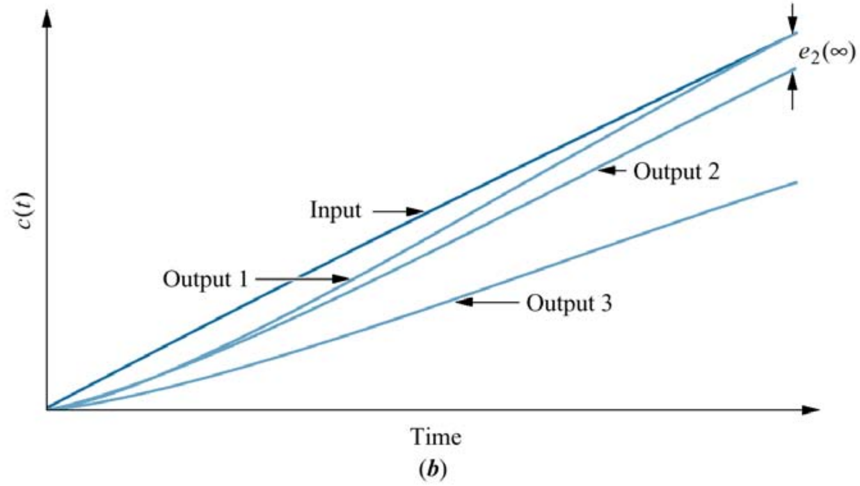
$$\omega_n = \sqrt{K}; \quad 2\zeta\omega_n = 5 \quad \Rightarrow \zeta = \frac{5}{2\sqrt{K}}.$$

For 10% overshoot or less, we need $\zeta = 0.591$ or more. Therefore,

$$K = 17.9 \quad \text{or less.}$$

Generalizing: steady-state error for arbitrary input





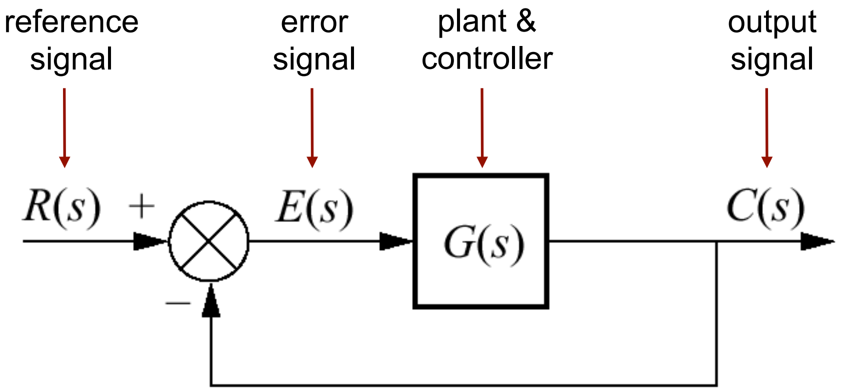
- Unit step input:
  Steady-state error =
  unit step – output as t→∞
- Ramp input:
  Steady-state error =
  ramp – output as t→∞

Generally, the steady–state error is defined as

$$e(\infty) = \lim_{t\to\infty} \Big[r(t) - c(t)\Big] = \lim_{s\to 0} s \Big[R(s) - C(s)\Big],$$

where the last equality follows from the final value theorem.

Generalizing: steady-state error for arbitrary system, unity feedback



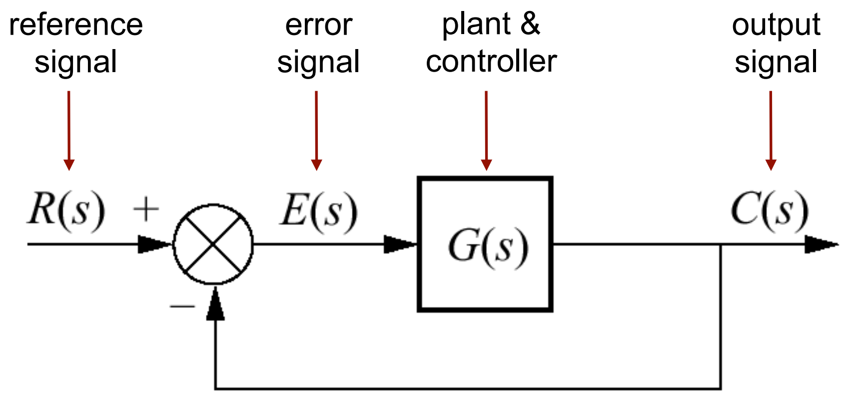
From the definition of the steady–state error,

$$e(\infty) = \lim_{s\to 0} s \left[R(s) - C(s)\right] = \lim_{s\to 0} sE(s).$$

From the block diagram we can also see that

$$\frac{C(s)}{E(s)} = G(s) \Rightarrow E(s) = \frac{C(s)}{G(s)}.$$

Generalizing: steady-state error for arbitrary system, unity feedback



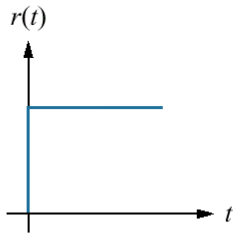
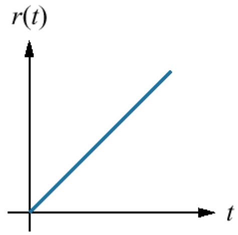
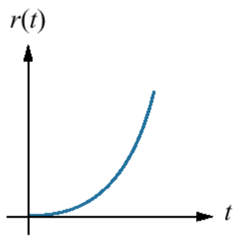
Recall the closed–loop TF of the unity feedback system

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} \Rightarrow C(s) = \frac{R(s)G(s)}{1+G(s)}.$$

Substituting into the two formulae from the previous page,

$$E(s) = \frac{R(s)}{1+G(s)} \Rightarrow e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}.$$

Steady-state error and static error constants

Waveform	Name	Physical interpretation	Time function	Laplace transform
	Step	Constant position	1	$\frac{1}{s}$
	Ramp	Constant velocity	t	$\frac{1}{s^2}$
	Parabola	Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

$$e(\infty) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}.$$

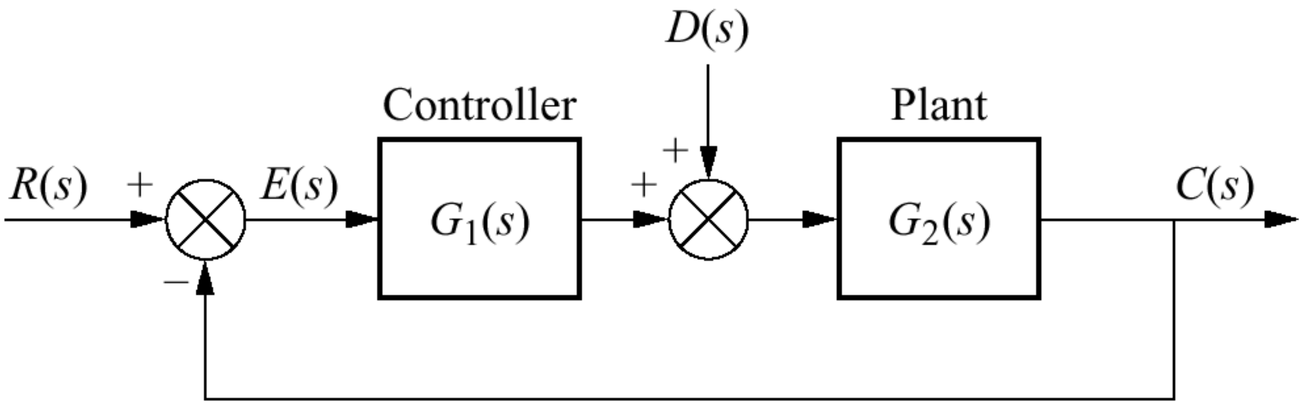
$$\begin{aligned} e(\infty) &= \lim_{s\to 0} \tfrac{1}{s} \times \tfrac{s}{1+G(s)} \\ &= \lim_{s\to 0} \tfrac{1}{1+G(s)} \\ &= \tfrac{1}{1+\lim_{s\to 0} G(s)} \equiv \tfrac{1}{1+K_p} \\ \text{where} \quad K_p &= \lim_{s\to 0} G(s). \end{aligned}$$

$$\begin{aligned} e(\infty) &= \lim_{s\to 0} \tfrac{1}{s^2} \times \tfrac{s}{1+G(s)} \\ &= \lim_{s\to 0} \tfrac{1}{s+sG(s)} \\ &= \tfrac{1}{\lim_{s\to 0} sG(s)} \equiv \tfrac{1}{1+K_v} \\ \text{where} \quad K_v &= \lim_{s\to 0} sG(s). \end{aligned}$$

$$\begin{aligned} e(\infty) &= \lim_{s\to 0} \tfrac{1}{s^3} \times \tfrac{s}{1+G(s)} \\ &= \lim_{s\to 0} \tfrac{1}{s^2+s^2G(s)} \\ &= \tfrac{1}{\lim_{s\to 0} s^2G(s)} \equiv \tfrac{1}{1+K_a} \\ \text{where} \quad K_a &= \lim_{s\to 0} s^2G(s). \end{aligned}$$

Note: the system must be **stable** (*i.e.*, all poles on left-hand side or at the origin) for these calculations to apply

Disturbances



From the I–O relationship of the plant,

$$C(s) = E(s)G_1(s)G_2(s) + D(s)G_2(s).$$

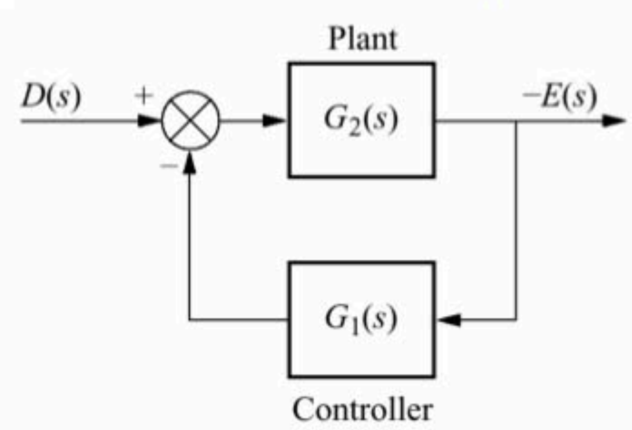
From the summation element,

$$E(s) = R(s) - C(s).$$

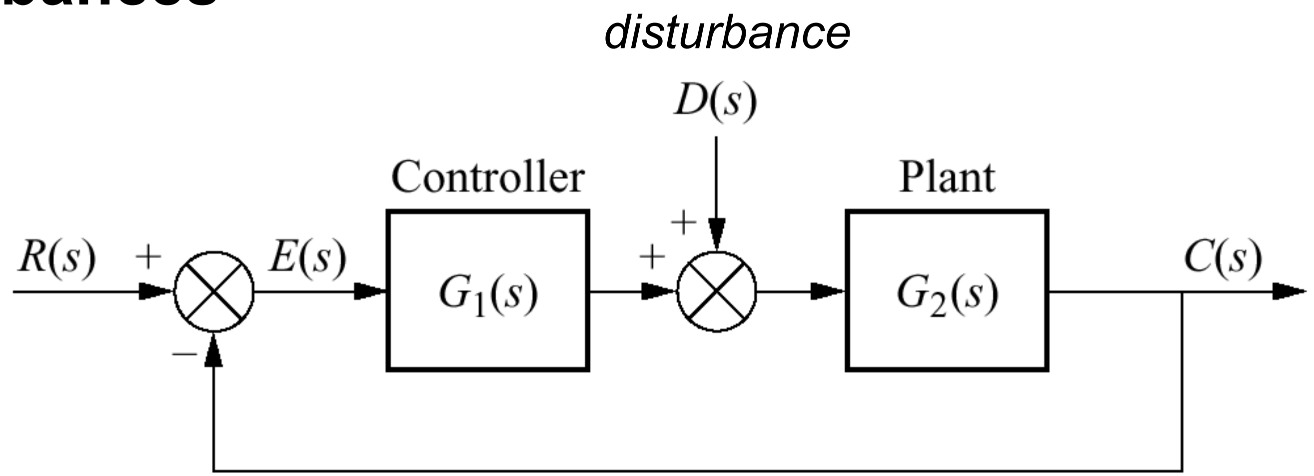
Substituting $C(s)$ and solving for $E(s)$,

$$E(s) = R(s)\frac{1}{1+G_1(s)G_2(s)} - D(s)\frac{G_2(s)}{1+G_1(s)G_2(s)}$$

Equivalent block diagram with $D(s)$ as input and $-E(s)$ as output.



Disturbances



$$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\left[R(s)\frac{1}{1+G_1(s)G_2(s)} - D(s)\frac{G_2(s)}{1+G_1(s)G_2(s)}\right] \equiv e_R(\infty)+e_D(\infty),$$

where

$$e_R(\infty) = \lim_{s\to 0}\frac{sR(s)}{1+G_1(s)G_2(s)}$$

$$e_D(\infty) = -\lim_{s\to 0}\frac{sG_2(s)D(s)}{1+G_1(s)G_2(s)}.$$