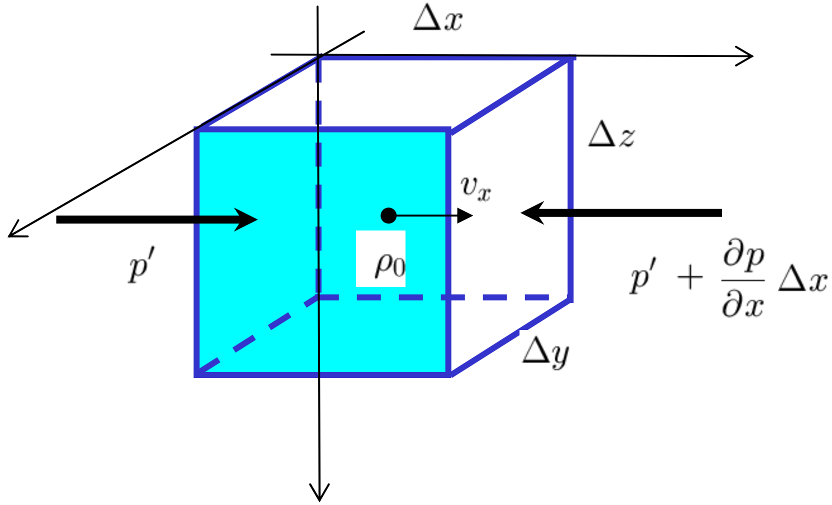
13.811
Advanced Structural Dynamics and Acoustics

The Acoustic Equation of Motion



Perturbation

$$
\begin{aligned}
p &= p_0 + p' \\
\rho &= \rho_0 + \rho'
\end{aligned}
$$

1D equation of motion

$$
\begin{aligned}
\rho_0 \Delta x \Delta y \Delta z \frac{\partial v_x}{\partial t} &= -\left(\frac{\partial p}{\partial x}\Delta x\right)\Delta y \Delta z \\
\Rightarrow \rho_0 \frac{\partial v_x}{\partial t} &= -\frac{\partial p}{\partial x}
\end{aligned}
$$

Other Coordinates

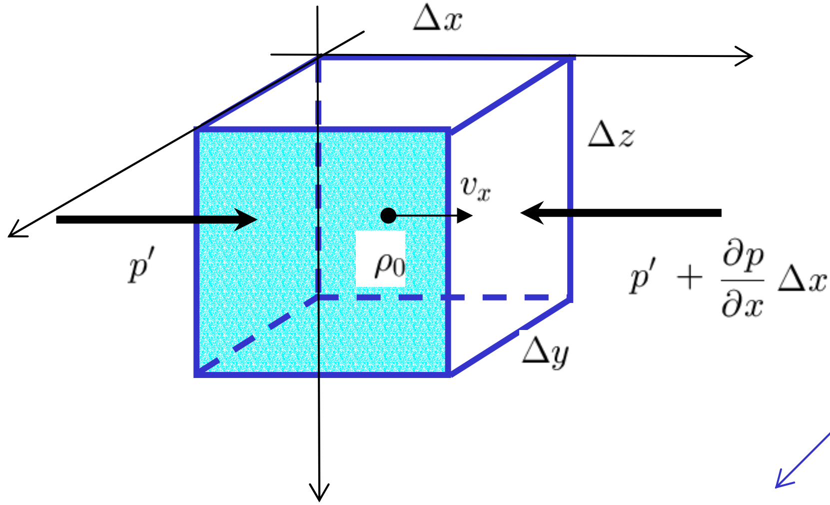
$$
\begin{aligned}
\rho_0 \frac{\partial v_y}{\partial t} &= -\frac{\partial p}{\partial y} \\
\rho_0 \frac{\partial v_z}{\partial t} &= -\frac{\partial p}{\partial z}
\end{aligned}
$$

Equation of Motion

$$
\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\overline{\nabla} p'
$$

$$
\overline{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)
$$

The Acoustic Wave Equation



Conservation of Mass

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \mathbf{v}$$

Equation of Motion

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\overline{\nabla} p'$$

Constitutive Equation

$$p = p_0 + \rho' \left[\frac{\partial p}{\partial \rho}\right]_S + \frac{1}{2} (\rho')^2 \left[\frac{\partial^2 p}{\partial \rho^2}\right]_S + \cdots$$

Speed of Sound

$$c^2 \equiv \left[\frac{\partial p}{\partial \rho}\right]_S \quad \text{(Sound speed)}$$

Linearly Elastic Fluid

$$\begin{aligned} p &= p_0 + p' \\ \rho &= \rho_0 + \rho' \\ p' &= \rho' c^2 . \end{aligned}$$

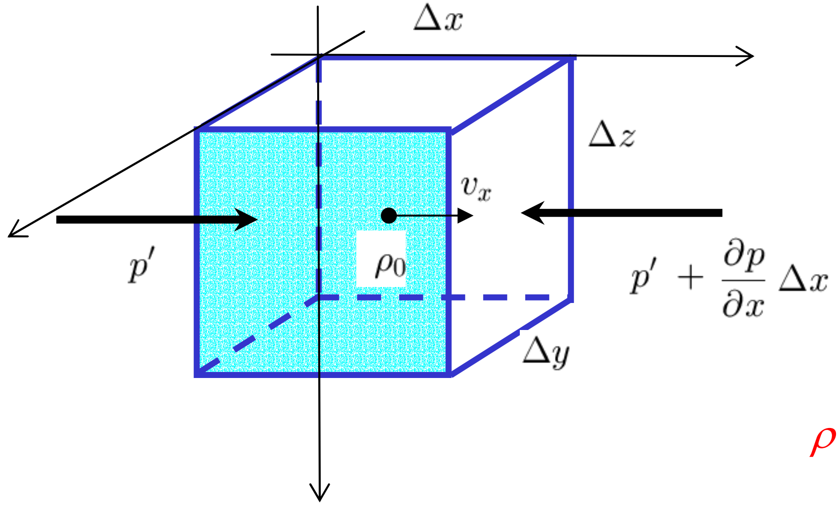
Pressure Wave Equation

$$\rho \nabla \cdot \left(\frac{1}{\rho} \nabla p\right) - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0 ,$$

Particle Velocity Wave Equation

$$\frac{1}{\rho} \nabla \left(\rho c^2 \nabla \cdot \mathbf{v}\right) - \frac{\partial^2 \mathbf{v}}{\partial t^2} = \mathbf{0} .$$

Potential Wave Equations



ρ constant

Pressure Wave Equation

$$\rho \nabla \cdot \left(\frac{1}{\rho} \nabla p \right) - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0 \,,$$

Particle Velocity Wave Equation

$$\frac{1}{\rho} \nabla \left(\rho c^2 \, \nabla \cdot \mathbf{v} \right) - \frac{\partial^2 \mathbf{v}}{\partial t^2} = \mathbf{0} \,.$$

Wavefield Potentials

Velocity Potential

$$\mathbf{v} = \nabla \phi \,.$$

$$\nabla \left(c^2 \, \nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} \right) = \mathbf{0} \,.$$

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = 0 \,,$$

Displacement Potential

$$\mathbf{u} = \nabla \psi \,,$$

$$\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0 \,.$$

$$p = -K \, \nabla^2 \psi \,,$$

$$K = \rho c^2 \,.$$

$$p = -\rho \frac{\partial^2 \psi}{\partial t^2} \,.$$

Solution of the Wave Equations

- 4-D Partial Differential Equation
- Analytical solutions only for few canonical problems
- Direct Numerical Solution (FDM, FEM)
  - Computationally intensive ($\Delta x \ll \lambda\,,\ \Delta t \ll T$).
- Dimension Reduction for PDE
  - Geometrical symmetries (Plane or axisymmetric problems)
  - Integral transforms
  - Analytical or numerical solution of ODE or low dimensional PDE.
  - Evaluation of inverse transforms (analytical or numerical)

Helmholtz Equation

Frequency-Time Fourier Transform

$$
\begin{aligned}
f(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega)\, e^{-i\omega t} d\omega \,, \\
f(\omega) &= \int_{-\infty}^{\infty} f(t)\, e^{i\omega t} dt \,,
\end{aligned}
$$

Helmholtz Equation

$$
\left[\nabla^2 + k^2(\mathbf{r})\right] \psi(\mathbf{r},\omega) = 0 \,,
$$

$$
k(\mathbf{r}) = \frac{\omega}{c(\mathbf{r})} \,.
$$

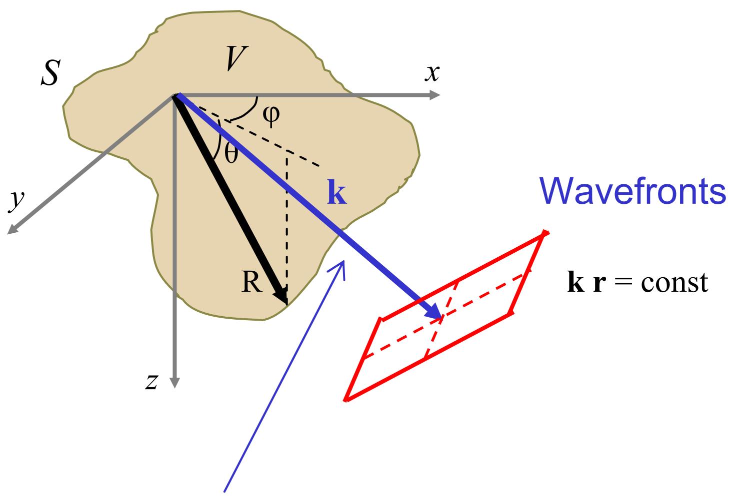
Solution of Helmholtz Equation

- Dimensionality of the problem.
- Medium wavenumber variation $k(\mathbf{r})$, i.e., the sound speed variation $c(\mathbf{r})$.
- Boundary conditions.
- Source–receiver geometry.
- Frequency and bandwidth.

Helmholtz Equation
Homogeneous Media

Helmholts Equation

$$\left[\nabla^2 + k^2\right] \psi(\mathbf{r}, \omega) = 0 \, ,$$



Wavevector **k** determines propagation direction of plane waves

Laplacian in Cartesian Coordinates

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \, ,$$

Plane Waves

$$\psi(x, y, z) = \begin{cases} A\, e^{i\mathbf{k}\cdot\mathbf{r}} \\ B\, e^{-i\mathbf{k}\cdot\mathbf{r}} \, , \end{cases}$$

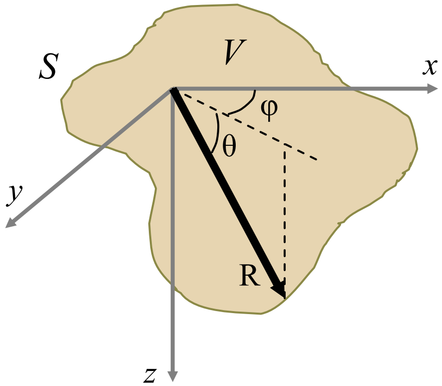
Wavevector

$$\mathbf{k} = (k_x, k_y, k_z)$$
$$|\mathbf{k}| = k = \omega/c$$

1-D propagation: $k_y, k_z = 0$:

$$\psi(x) = \begin{cases} A\, e^{ikx} & \text{Forward propagating} \\ B\, e^{-ikx} & \text{Backward propagating} \end{cases}$$

Helmholtz Equation
Homogeneous Media



Cylindrical Cordinates

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} .$$

Axial Symmetry

$$\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + k^2\right]\psi(r) = 0 ,$$

bessel Functions

$$\psi(r) = \begin{cases} A\, J_0(kr) \\ B\, Y_0(kr) , \end{cases}$$

Hankel Functions

$$\psi(r) = \begin{cases} CH_0^{(1)}(kr) & = & C\,[J_0(kr) + iY_0(kr)] \\ DH_0^{(2)}(kr) & = & D\,[J_0(kr) - iY_0(kr)] . \end{cases}$$

$$H_0^{(1)}(kr) \simeq \sqrt{\frac{2}{\pi kr}}\, e^{i(kr-\pi/4)} \text{ Diverging waves}$$

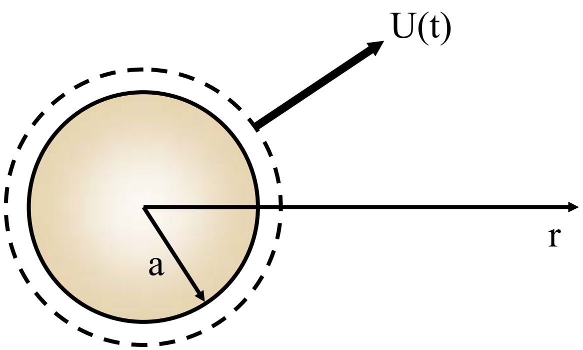
$$H_0^{(2)}(kr) \simeq \sqrt{\frac{2}{\pi kr}}\, e^{-i(kr-\pi/4)} \text{ Converging waves}$$

Spherical Cordinates

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + k^2\right]\psi(r) = 0 ,$$

$$\psi(r) = \begin{cases} (A/r)\, e^{ikr} & \text{Diverging waves} \\ (B/r)\, e^{-ikr} & \text{Converging waves} \end{cases}$$

Radiation of Sound
The Point Source



Unbounded Homogeneous Medium

Frequency Domain

$$u_r(a) = U(\omega)\,.$$

Spherical geometry solution

$$\psi(r) = A\,\frac{e^{ikr}}{r}\,,$$

$$u_r(r) = \frac{\partial \psi(r)}{\partial r} = A\,e^{ikr}\left(\frac{ik}{r} - \frac{1}{r^2}\right).$$

Simple Point Source

$$ka \ll 1$$

$$u_r(a) = A\,e^{ika}\,\frac{ika-1}{a^2} \simeq -\frac{A}{a^2}\,,$$

$$A = -a^2\,U(\omega)\,.$$

$$\Rightarrow$$

$$\psi(r) = -S_\omega\,\frac{e^{ikr}}{4\pi r}\,.$$

$$S_\omega = 4\pi a^2\,U(\omega)$$

Green's Function

$$g_\omega(r,0) = \frac{e^{ikr}}{4\pi r} \,,$$

Source at r_0

$$g_\omega(\mathbf{r},\mathbf{r}_0) = \frac{e^{ikR}}{4\pi R}\,, \quad R = |\mathbf{r}-\mathbf{r}_0| \,.$$

Helmholtz Equation for Green's function

$$\left[\nabla^2 + k^2\right] g_\omega(\mathbf{r},\mathbf{r}_0) = -\delta(\mathbf{r}-\mathbf{r}_0) \,,$$

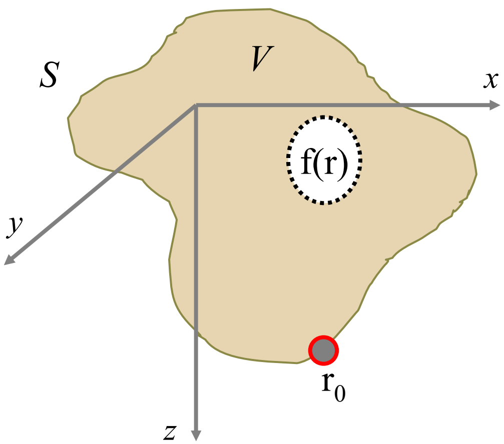
Integrate over spherical volume V of radius $\epsilon \to 0$:

$$\begin{aligned}
\int_V -\delta(\mathbf{r}-\mathbf{r}_0)dV &= -1 \\
\int_V k^2 g_\omega(\mathbf{r},\mathbf{r}_0)dV &\to_{\epsilon\to 0} 0 \\
\int_V \nabla^2 g_\omega(\mathbf{r},\mathbf{r}_0)dV &= \int_S \frac{\partial}{\partial R} g_\omega(\mathbf{r},\mathbf{r}_0)dS \\
&= \int_S \left[-\frac{e^{ik\epsilon}}{4\pi\epsilon^2} + \frac{ike^{ik\epsilon}}{4\pi\epsilon}\right] dS \\
&= 4\pi\epsilon^2 \left[-\frac{e^{ik\epsilon}}{4\pi\epsilon^2} + \frac{ike^{ik\epsilon}}{4\pi\epsilon}\right] \to_{\epsilon\to 0} -1
\end{aligned}$$

Reciprocity

$$g_\omega(\mathbf{r},\mathbf{r}_0) = g_\omega(\mathbf{r}_0,\mathbf{r}) \,,$$

Green's Theorem



Source in Bounded Medium

Inhomogeneous Helmholtz Equation

$$\left[\nabla^2 + k^2\right] \psi(\mathbf{r}) = f(\mathbf{r})\ .$$

General Green's Function

$$G_\omega(\mathbf{r}, \mathbf{r}_0) = g_\omega(\mathbf{r}, \mathbf{r}_0) + H_\omega(\mathbf{r})\ ,$$

$$\left[\nabla^2 + k^2\right] H_\omega(\mathbf{r}) = 0\ .$$

$$\Rightarrow$$

$$\left[\nabla^2 + k^2\right] G_\omega(\mathbf{r}, \mathbf{r}_0) \quad = \quad -\delta(\mathbf{r} - \mathbf{r}_0)\ .$$

Green's Theorem

$$\psi(\mathbf{r}) = \int_S \left[G_\omega(\mathbf{r}, \mathbf{r}_0) \frac{\partial \psi(\mathbf{r}_0)}{\partial \mathbf{n}_0} - \psi(\mathbf{r}_0) \frac{\partial G_\omega(\mathbf{r}, \mathbf{r}_0)}{\partial \mathbf{n}_0} \right] dS_0 - \int_V f(\mathbf{r}_0)\, G_\omega(\mathbf{r}, \mathbf{r}_0)\, dV_0,$$

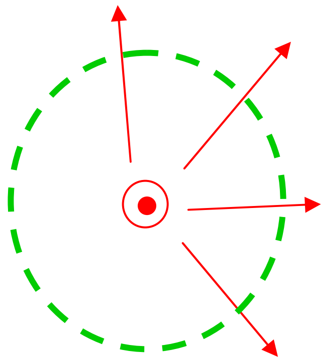
Source in Infinite Medium

$$\psi(\mathbf{r}) = -\int_V f(\mathbf{r}_0)\, g_\omega(\mathbf{r},\mathbf{r}_0)\, dV_0 \,.$$

For any imaginary surface enclosing the sources:

$$\int_S \left[g_\omega(\mathbf{r},\mathbf{r}_0) \frac{\partial \psi(\mathbf{r}_0)}{\partial \mathbf{n}_0} - \psi(\mathbf{r}_0) \frac{\partial g_\omega(\mathbf{r},\mathbf{r}_0)}{\partial \mathbf{n}_0} \right] dS_0 = 0 \,.$$

$$\Rightarrow \quad \int_S \frac{e^{ikR}}{4\pi R} \left[\frac{\partial \psi(\mathbf{r}_0)}{\partial R} - ik\,\psi(\mathbf{r}_0) \right] dS_0 = 0 \,.$$



Radiation Condition

$$R \left[\frac{\partial}{\partial R} - ik \right] \psi(\mathbf{r}_0) \to 0 \,, \quad R = |\mathbf{r} - \mathbf{r}_0| \to \infty \,.$$