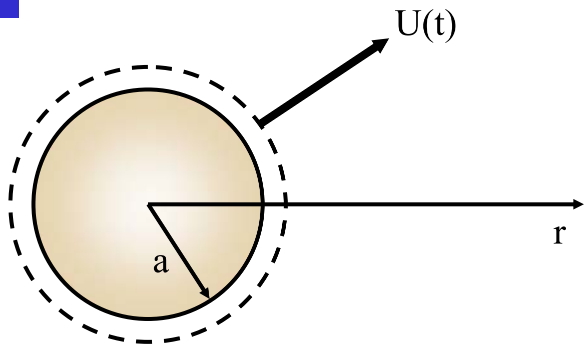
13.811
Advanced Structural Dynamics and Acoustics

Acoustics
Lecture 4

Green’s Function



Helmholts Equation for Green’s Function

$$\left[\nabla^2 + k^2\right] g_\omega(\mathbf{r}, \mathbf{r}_0) = -\delta(\mathbf{r} - \mathbf{r}_0) \; ,$$

$$\psi(r) = -S_\omega \frac{e^{ikr}}{4\pi r} .$$

$$S_\omega = 4\pi a^2 U(\omega)$$

Green’s Function

,

$$g_\omega(r, 0) = \frac{e^{ikr}}{4\pi r} \; ,$$

Source at r_0

$$g_\omega(\mathbf{r}, \mathbf{r}_0) = \frac{e^{ikR}}{4\pi R} \; , \quad R = |\mathbf{r} - \mathbf{r}_0| \; .$$

Proof

$$\int_V -\delta(\mathbf{r} - \mathbf{r}_0) dV = -1$$

$$\int_V k^2 g_\omega(\mathbf{r}, \mathbf{r}_0) dV \rightarrow_{\epsilon \to 0} 0$$

$$\int_V \nabla^2 g_\omega(\mathbf{r}, \mathbf{r}_0) dV = \int_S \frac{\partial}{\partial R} g_\omega(\mathbf{r}, \mathbf{r}_0) dS$$

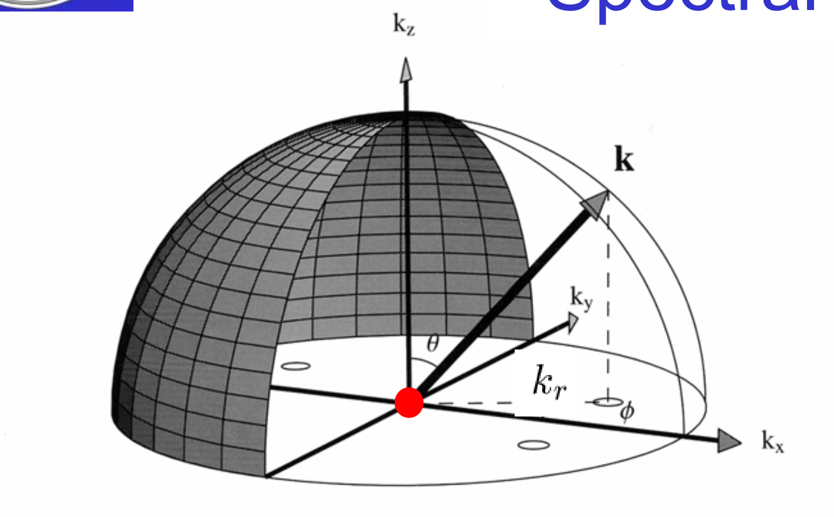
$$= \int_S \left[-\frac{e^{ik\epsilon}}{4\pi\epsilon^2} + \frac{ike^{ik\epsilon}}{4\pi\epsilon} \right] dS$$

$$= 4\pi\epsilon^2 \left[-\frac{e^{ik\epsilon}}{4\pi\epsilon^2} + \frac{ike^{ik\epsilon}}{4\pi\epsilon} \right] \rightarrow_{\epsilon \to 0} -1$$

Reciprocity

$$g_\omega(\mathbf{r}, \mathbf{r}_0) = g_\omega(\mathbf{r}_0, \mathbf{r}) \; ,$$

Green's Function
Spectral Representations



Cylindrical Coordinates

$$g_\omega = \frac{e^{ikR}}{4\pi r} = \frac{i}{4\pi}\int_0^\infty \frac{e^{ik_z|z-z_0|}}{k_z} k_r J_0(k_r r) dk_r$$

$$k_r^2 = k_x^2 + k_y^2$$

$$k_z = \begin{cases} \sqrt{k^2 - k_r^2}\,, & k_r \le k \\ i\sqrt{k_r^2 - k^2}\,, & k_r > k\,. \end{cases}$$

$$k_r = k\sin\theta$$

$$k_x = k\sin\theta cos\phi$$

$$k_y = k\sin\theta sin\phi$$

Radiating Spectrum

$$k|z - z_0| >> 1$$

$$g_\omega = \frac{e^{ikR}}{4\pi R} \simeq \frac{i}{4\pi}\int_0^k \frac{e^{ik_z|z-z_0|}}{k_z} k_r J_0(k_r r) dk_r$$

$$k_r = k\sin\theta\,;\quad dk_r = k\cos\theta d\theta = k_z d\theta$$

Normalized Pressure – $p(1m) = 1$

$$p_\omega = \frac{e^{ikR}}{R} = i\int_0^{\frac{\pi}{2}} e^{ik|z-z_0|\cos\theta} k_r J_0(k_r r) d\theta$$

Far Field Directivity Function

$$kR >> 1$$

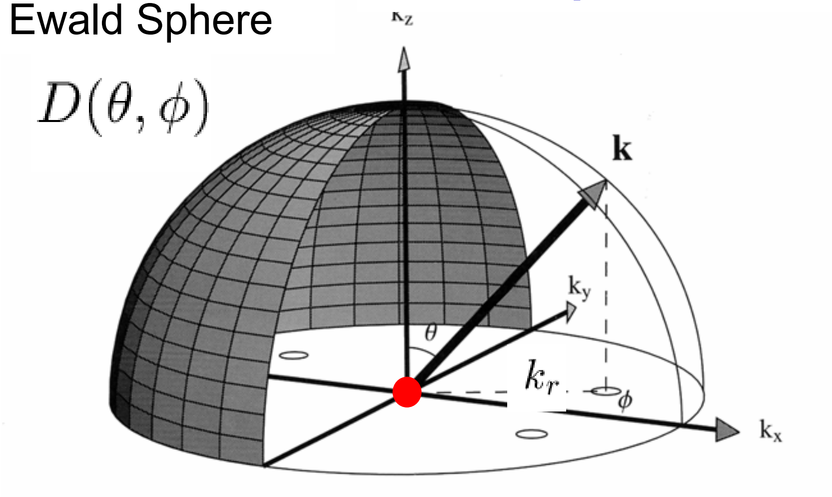
$$p_\omega(R,\theta) = \boxed{D(\theta)}\frac{e^{ikR}}{R} \simeq i\int_0^{\frac{\pi}{2}} \boxed{D(\theta)}\left[e^{ik|z-z_0|\cos\theta} k_r J_0(k_r r)\right] d\theta$$

Simple Point Source

$$D(\theta) = 1$$

Green's Function
Spectral Representations

Ewald Sphere



Cartesian Coordinates

$$g_\omega = \frac{e^{ikR}}{4\pi R} = \frac{i}{8\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{ik_z|z-z_0|}}{k_z}e^{ik_x x}e^{ik_y y}dk_x dk_y$$

$$k_r^2 = k_x^2 + k_y^2$$

$$k_z = \begin{cases} \sqrt{k^2 - k_r^2}\,, & k_r \le k \\ i\sqrt{k_r^2 - k^2}\,, & k_r > k\,. \end{cases}$$

$$k_r = k\sin\theta$$

$$k_x = k\sin\theta cos\phi$$

$$k_y = k\sin\theta sin\phi$$

Normalized Pressure – *p(1m) = 1*

$$p_\omega = 4\pi g_\omega = \frac{e^{ikR}}{R} = \frac{i}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{ik_z|z-z_0|}}{k_z}e^{ik_x x}e^{ik_y y}dk_x dk_y$$

Far Field Directivity Function

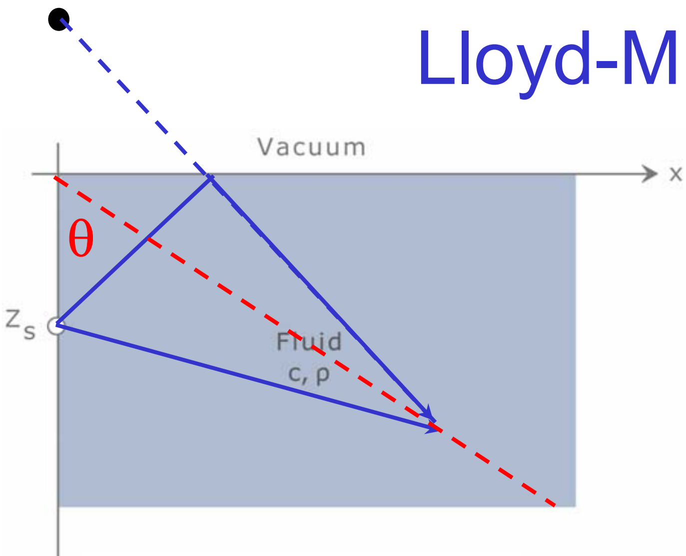
$$kR >> 1$$

$$p_\omega = \boxed{D(\theta,\phi)}\frac{e^{ikR}}{R} \simeq \frac{i}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\boxed{D(\theta,\phi)}\frac{e^{ik_z|z-z_0|}}{k_z}e^{ik_x x}e^{ik_y y}dk_x dk_y$$

Simple Point Source

$$D(\theta,\phi) = 1$$

Directivity Function
Lloyd-Mirror Pattern



Directivity Function

$$D(\theta) = e^{-ikz_s\cos\theta} - e^{ikz_s\cos\theta}$$

Directivity Maxima and Minima

$$2k_z z_s = (2n+1)\pi$$

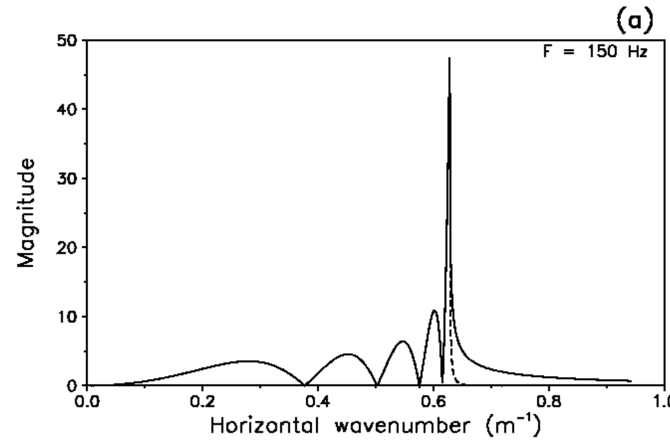
$$\Leftrightarrow$$

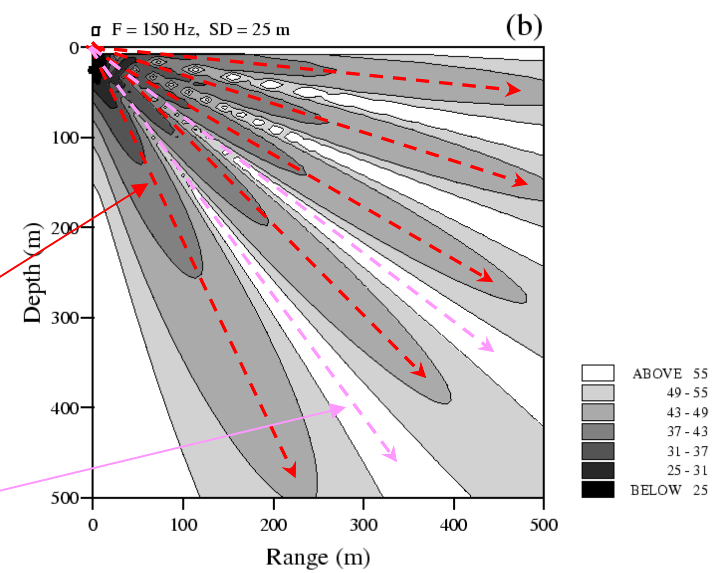
$$k_z = \frac{(2n+1)\pi}{2z_s}$$

$$\Leftrightarrow$$

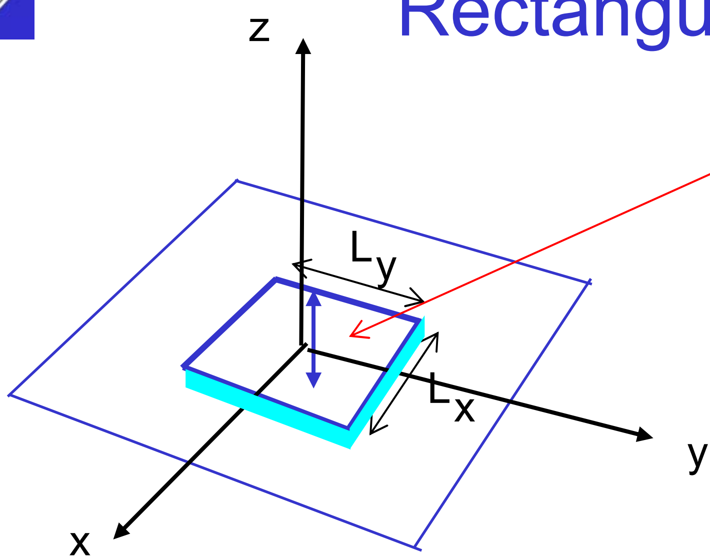
$$cos\theta_{\max} = \frac{(2n+1)\pi}{2kz_s}$$

$$cos\theta_{\min} = \frac{2n\pi}{2kz_s}$$





Directivity Function
Rectangular Baffled Piston



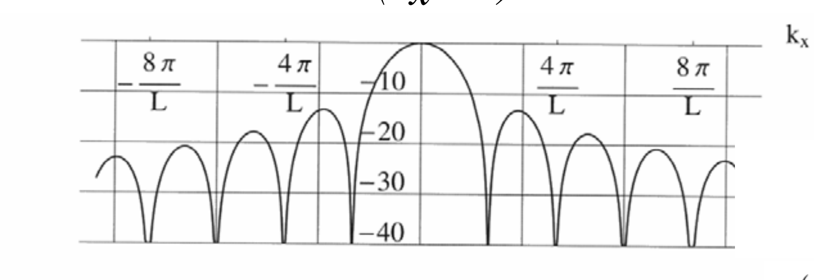
$$\dot{w}_\omega(x,y,0) = \Pi(x/L_x)\Pi(y/L_y) = \begin{cases} 1 & |x| < L_x/2,\ |y| < L_y/2 \\ 0 & \text{otherwize} \end{cases}$$

Fourier Transform

$$\int_{-\infty}^{\infty} \Pi(x/L) e^{-i(k_x x)} dx = L \mathrm{sinc}(k_x L/2)$$

$$\begin{aligned} \dot{w}_\omega(k_x, k_y; 0) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \dot{w}_\omega(x,y,0) e^{-i(k_x x + k_y y)} dx dy \\ &= L_x L_y \mathrm{sinc}(\frac{k_x L_x}{2}) \mathrm{sinc}(\frac{k_y L_y}{2}) \end{aligned}$$

sinc$(k_x L/2)$



Radiated Field

$$\begin{aligned} \psi_\omega(k_x, k_y; z) &= A(k_x, k_y) e^{ik_z z} \\ \dot{w}_\omega(k_x, k_y; z) &= -i\omega \frac{\partial \psi_\omega(k_x, k_y; z)}{\partial z} = \omega k_z A(k_x, k_y) e^{ik_z z} \end{aligned}$$

$$A(k_x, k_y) = \frac{L_x L_y}{k_z \omega} \mathrm{sinc}(\frac{k_x L_x}{2}) \mathrm{sinc}(\frac{k_y L_y}{2})$$

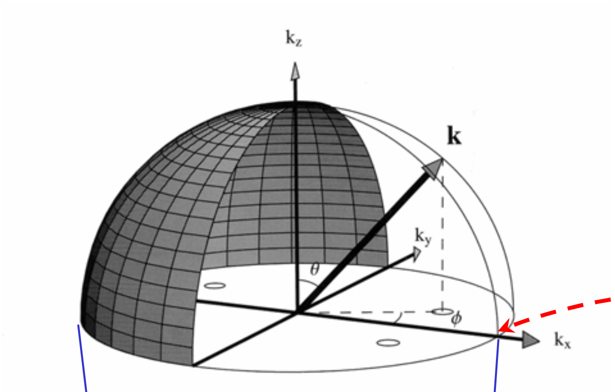
$$\begin{aligned} p_\omega(x,y,z) &= \rho\omega^2 \psi_\omega(x,y,z) \\ &= \frac{\rho\omega L_x L_y}{4\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{sinc}(\frac{k_x L_x}{2}) \mathrm{sinc}(\frac{k_y L_y}{2}) \frac{e^{ik_z|z-z_0|}}{k_z} e^{ik_x x} e^{ik_y y} dk_x dk_y \end{aligned}$$

Directivity Function - Definition

$$p_\omega = D(\theta,\phi)\frac{e^{ikR}}{R} \simeq \frac{i}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} D(\theta,\phi) \frac{e^{ik_z|z-z_0|}}{k_z} e^{ik_x x} e^{ik_y y} dk_x dk_y$$

$$D(\theta,\phi) = -\frac{i\rho\omega L_x L_y}{2\pi} \mathrm{sinc}(\frac{k_x L_x}{2}) \mathrm{sinc}(\frac{k_y L_y}{2})$$

Ewald Sphere Construction
Square Baffled Piston

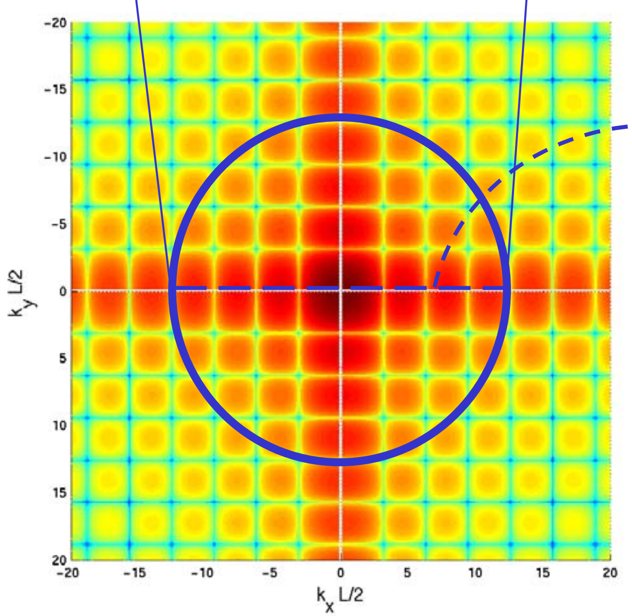


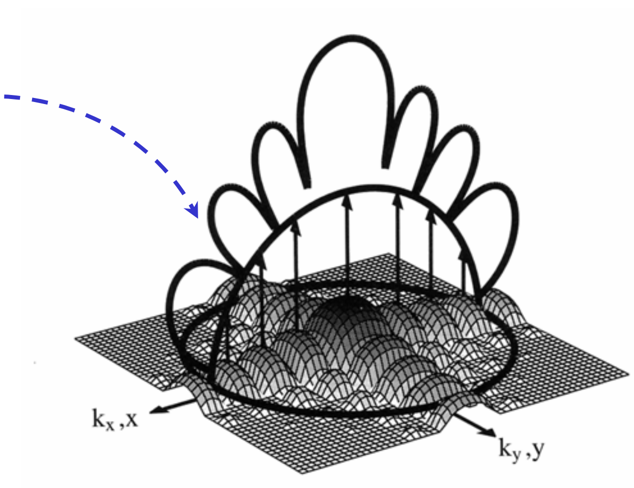
Directivity Function

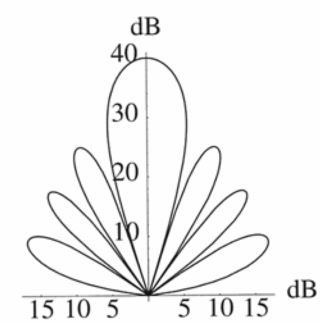
$$D(\theta,\phi) = -\frac{i\rho\omega L_x L_y}{2\pi}\mathrm{sinc}(\frac{k_x L_x}{2})\mathrm{sinc}(\frac{k_y L_y}{2})$$

$$L_y = L_x = L$$

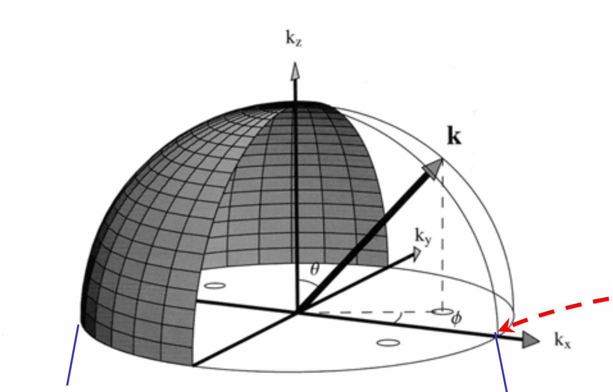
$$f = \omega/2\pi = kc/2\pi = 4c/L$$







Ewald Sphere Construction
Square Baffled Piston



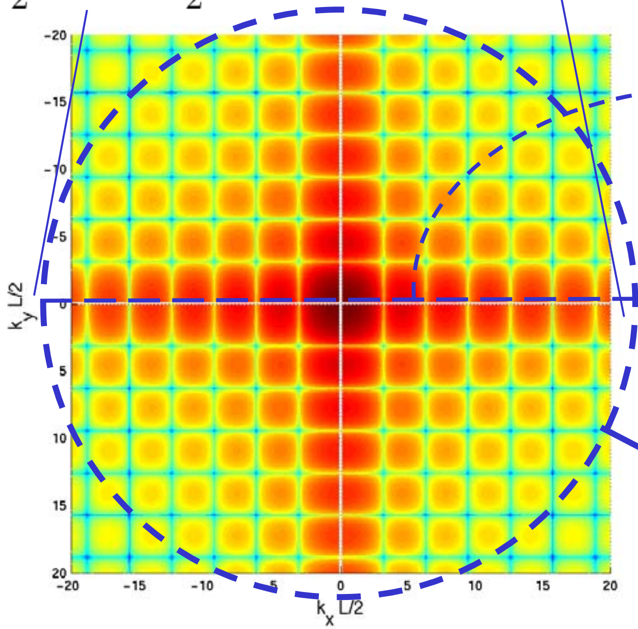
Directivity Function

$$D(\theta,\phi) = -\frac{i\rho\omega L_x L_y}{2\pi}\mathrm{sinc}(\frac{k_x L_x}{2})\mathrm{sinc}(\frac{k_y L_y}{2})$$

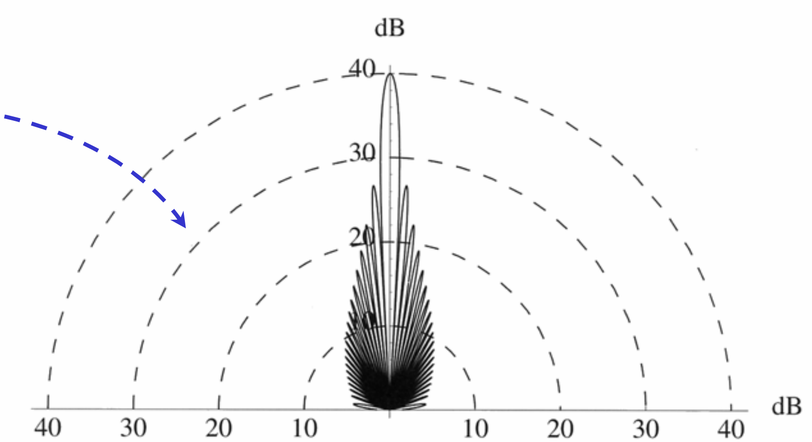
$$L_y = L_x = L$$

$$f = \omega/2\pi = kc/2\pi = 20c/L$$

$$\mathrm{sinc}(\frac{k_x L_x}{2})\mathrm{sinc}(\frac{k_y L_y}{2})$$



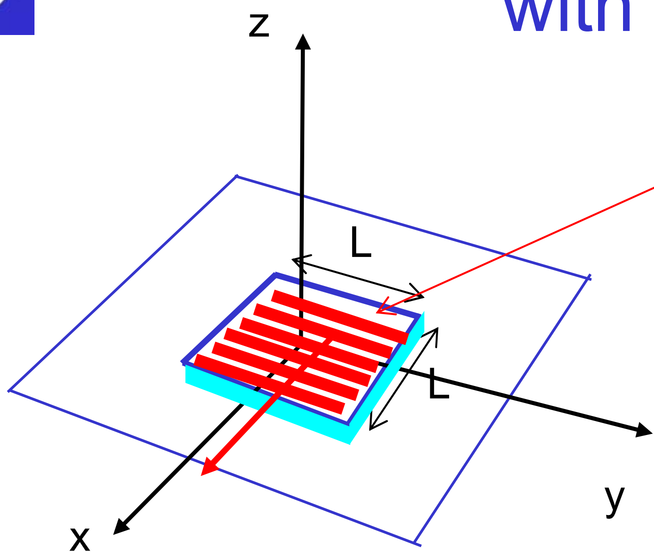
$$k = 40\pi/L$$



$$\lim_{L\to\infty} \mathrm{sinc}(k_x L/2) = \pi\delta(k_x)$$

$$\lim_{L_x,L_y\to\infty} D(\theta,\phi) = -\frac{i\pi\rho\omega\delta(k_x)\delta(k_y)}{2}$$

Square Baffled Plate with Traveling Wave



$$\dot{w}_\omega(x,y,0) = \begin{cases} e^{ik_{x0}x} & |x| < L/2,\ |y| < L/2 \\ 0 & \text{otherwize} \end{cases}$$

$$\dot{w}_\omega(x,y,0) = e^{ik_{x0}x}\Pi(x/L)\Pi(y/L)$$

$$\Pi(x/L) = \begin{cases} & |x| < L/2 \\ 0 & |x| > L/2 \end{cases}$$

$$\begin{aligned} \dot{w}_\omega(k_x,k_y;0) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \dot{w}_\omega(x,y,0)e^{-i(k_x x+k_y y)}dxdy \\ &= \int_{-\infty}^{\infty}\Pi(x/L)e^{ik_{x0}x}e^{-i(k_x x)}dx\int_{-\infty}^{\infty}\Pi(y/L)e^{-i(k_y y)}dy \\ &= L^2\left[\text{sinc}(k_x L/2) * \delta(k_x - k_{x0})\right]\text{sinc}(k_y L/2) \\ &= L^2\text{sinc}((k_x - k_{x0})L/2)\text{sinc}(k_y L/2) \end{aligned}$$

$$\int_{-\infty}^{\infty}\Pi(x/L)e^{-i(k_x x)}dx = L\text{sinc}(k_x L/2)$$

$$\int_{-\infty}^{\infty} e^{ik_{x0}x}e^{-ik_x x}dx = 2\pi\delta(k_x - k_{x0})$$

$$\int_{-\infty}^{\infty} f(x)g(x)e^{-ik_x x}dx = \frac{1}{2\pi}f(k_x) * g(k_x)$$

$$p_\omega(x,y,z) = \rho\omega^2\psi_\omega(x,y,z)$$

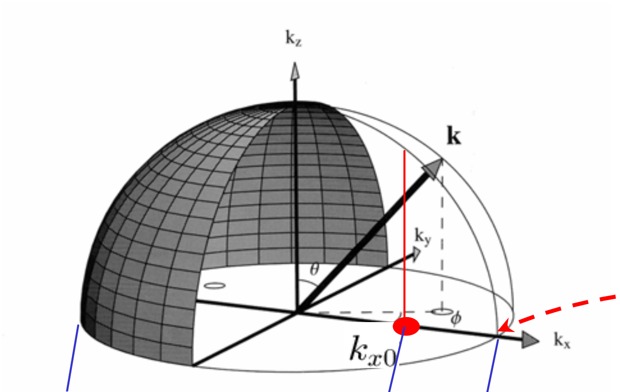
$$= \frac{\rho\omega L^2}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\text{sinc}\left(\frac{(k_x - k_{x0})L}{2}\right)\text{sinc}\left(\frac{k_y L}{2}\right)\frac{e^{ik_z|z-z_0|}}{k_z}e^{ik_x x}e^{ik_y y}dk_x dk_y$$

Directivity Function - Definition

$$p_\omega = D(\theta,\phi)\frac{e^{ikR}}{R} \simeq \frac{i}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} D(\theta,\phi)\frac{e^{ik_z|z-z_0|}}{k_z}e^{ik_x x}e^{ik_y y}dk_x dk_y$$

$$D(\theta,\phi) = -\frac{i\rho\omega L^2}{2\pi}\text{sinc}\left(\frac{(k_x - k_{x0})L}{2}\right)\text{sinc}\left(\frac{k_y L}{2}\right)$$

Square Baffled Plate with Traveling Wave – $\theta_0 = 30^o$



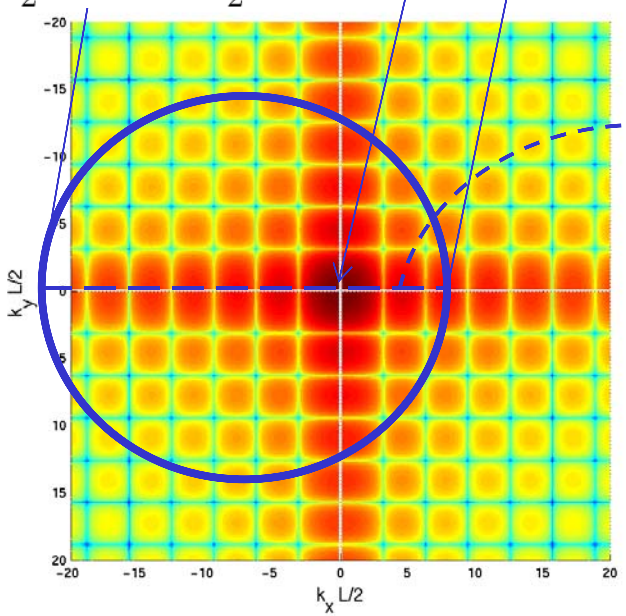
Directivity Function

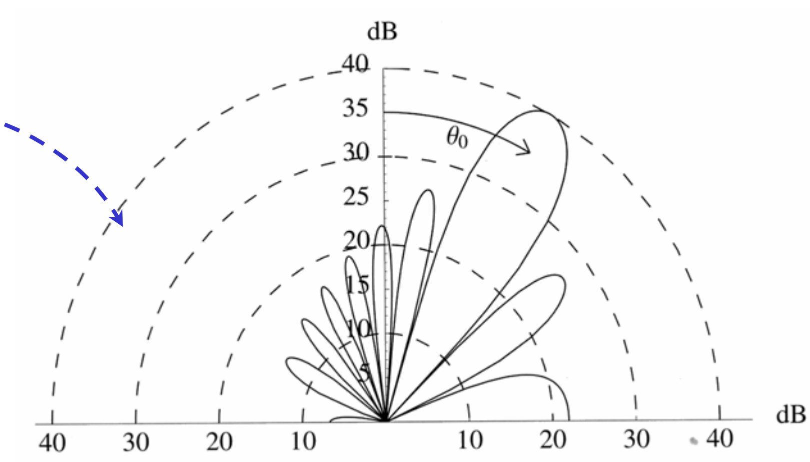
$$D(\theta,\phi) = -\frac{i\rho\omega L^2}{2\pi}\mathrm{sinc}(\frac{(k_x - k_{x0})L}{2})\mathrm{sinc}(\frac{k_y L}{2})$$

$$\mathrm{sinc}(k_x - k_{x0})L/2) = \mathrm{sinc}(kL(\sin\theta\cos\phi - sin\theta_0\cos\phi_0)/2)$$

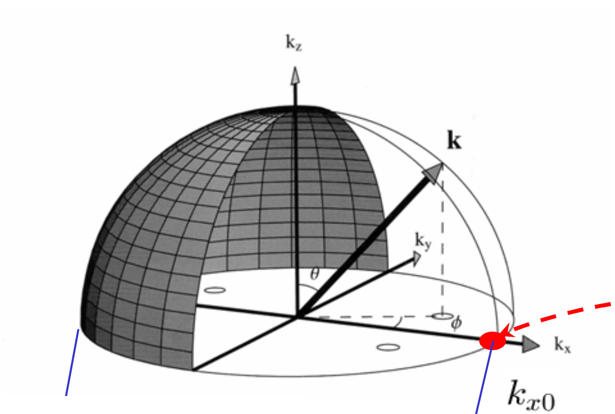
$$f = \omega/2\pi = kc/2\pi = 15c/\pi L$$

$$\mathrm{sinc}(\frac{(k_x - k_{x0})L}{2})\mathrm{sinc}(\frac{k_y L}{2})$$





Square Baffled Plate with Traveling Wave – $\theta_0 = 90^o$



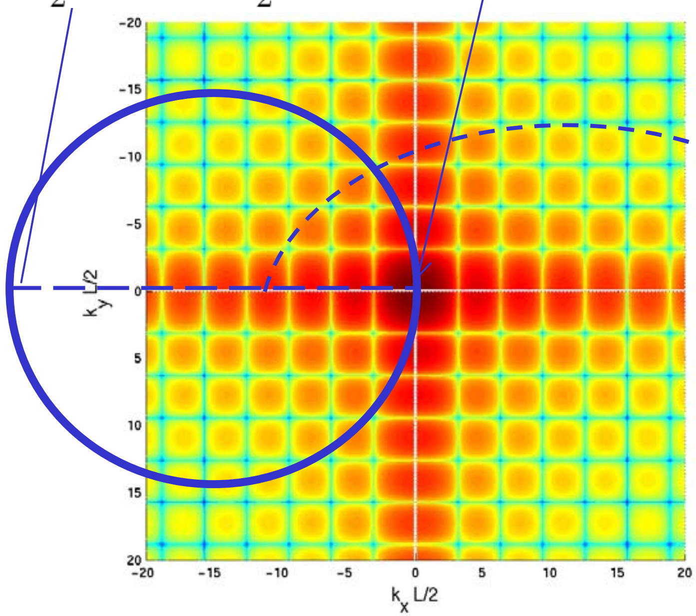
Directivity Function

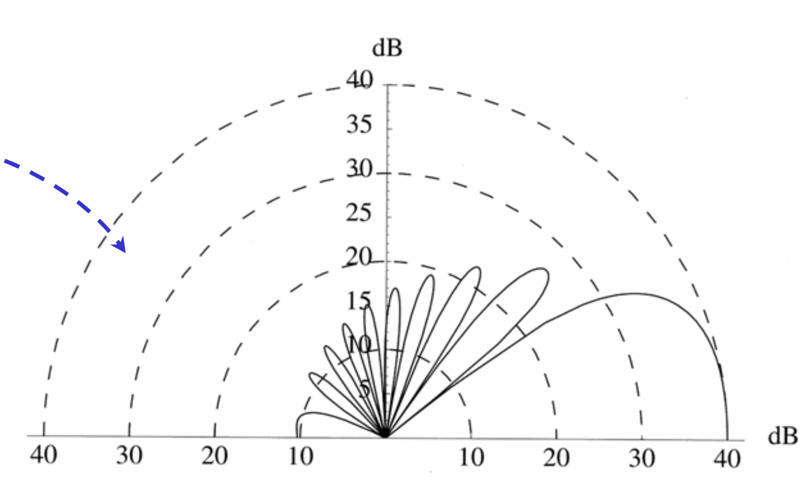
$$D(\theta,\phi) = -\frac{i\rho\omega L^2}{2\pi}\text{sinc}(\frac{(k_x - k_{x0})L}{2})\text{sinc}(\frac{k_y L}{2})$$

$$\text{sinc}((k_x - k_{x0})L/2) = \text{sinc}(kL(\sin\theta\cos\phi - sin\theta_0\cos\phi_0)/2)$$

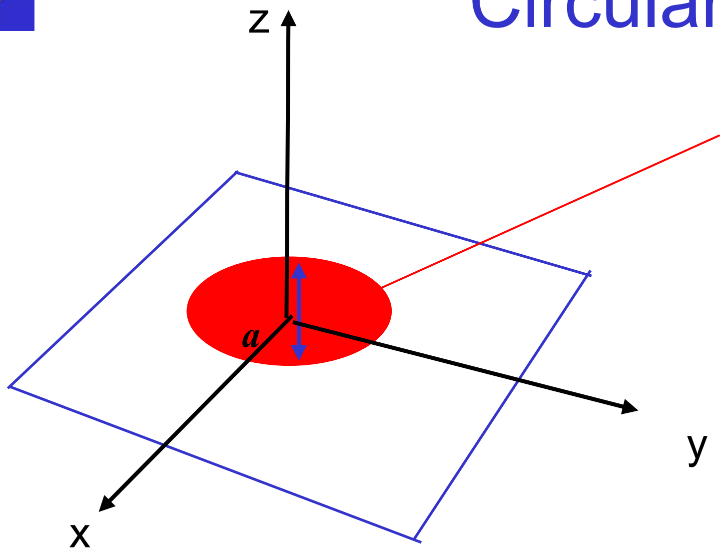
$$f = \omega/2\pi = kc/2\pi = 15c/\pi L$$

$$\text{sinc}(\frac{(k_x - k_{x0})L}{2})\text{sinc}(\frac{k_y L}{2})$$





Directivity Function
Circular Baffled Piston



$$\dot{w}_\omega(r,\phi,0) = \dot{w}_\omega(r,0) = \begin{cases} 1 & r \le a, \\ 0 & r > a \end{cases}$$

Hankel Transform

$$\begin{aligned} \dot{w}(k_r,0) &= \int_0^\infty \dot{w}(r,0) r J_0(k_r r) dr \\ &= \int_0^a J_0(k_r r) r dr = \frac{a}{k_r} J_1(k_r a) \end{aligned}$$

$$\dot{w}(k_r,z) = \dot{w}(k_r,0) e^{ik_z z}$$

$$\left.\begin{aligned} \dot{w}(k_r,z) &= \omega k_z \psi_\omega(k_r,z) \\ p_\omega(k_r,z) &= \rho\omega^2 \psi_\omega(k_r,z) \end{aligned}\right\} \Rightarrow p_\omega(k_r,z) = \frac{\rho\omega}{k_z} \dot{w}(k_r,z)$$

Radiated Pressure Field

$$p_\omega(r,z) = \rho\omega \int_0^a \frac{a}{k_r} J_1(k_r a) \frac{e^{ik_z z}}{k_z} J_0(k_r r) k_r dk_r$$

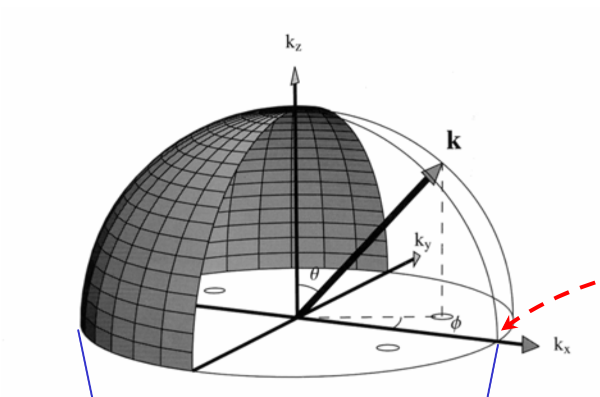
Directivity Function - Definition

$$p_\omega(r,z) = D(\theta) \frac{e^{ikR}}{R} \simeq i \int_0^k D(\theta) \frac{e^{ik_z|z-z_0|}}{k_z} k_r J_0(k_r r) dk_r$$

$$\begin{aligned} D(\theta,\phi) = D(\theta) &= -i\rho\omega a^2 \frac{J_1(k_r a)}{k_r a} \\ &= -\frac{i\rho\omega S_\omega}{\pi} \frac{J_1(k_r a)}{k_r a} \end{aligned}$$

Source Strength - Volume Injection $S_\omega = \pi a^2$

Ewald Sphere Construction
Circular Baffled Piston



Directivity Function

$$D(\theta,\phi) = D(\theta) \;=\; -i\rho\omega a^2 \frac{J_1(k_r a)}{k_r a}$$

$$f = \omega/2\pi = kc/2\pi = 5c/a\pi \quad (ka = 10)$$

$$\frac{J_1(k_r a)}{k_r a}$$