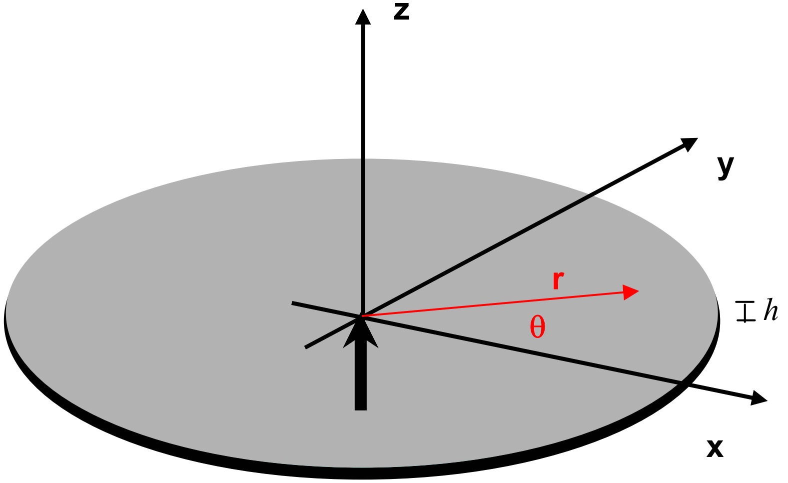
13.811
Advanced Structural Dynamics and Acoustics

Acoustics
Lecture 6

Point-Driven Plate Radiation



Cylindrical Coordinates

$$D\left(\frac{d^2}{dr^2}+\frac{d}{rdr}\right)^2 w_\omega - \rho_s h\omega^2 w_\omega = F_\omega \frac{\delta(r)}{2\pi r} - p_{\omega}(r)$$

Light Fluid Loading

$$Dk_r^4 w_\omega(k_r) - \rho_s h\omega^2 w_\omega(k_r) \simeq F_\omega$$

$$w_\omega(k_r) = \frac{F(\omega)}{2\pi D(k_r^4 - k_f^4)}$$

Flexural Wavenumber

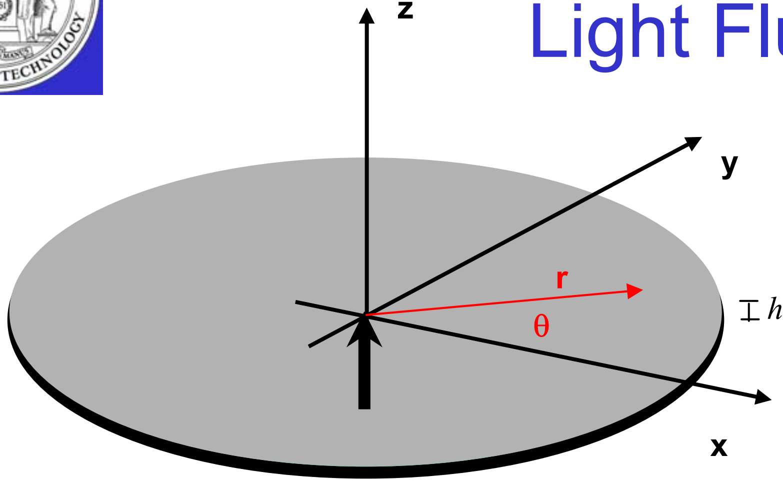
$$k_f = \left(\frac{m_s\omega^2}{D}\right)^{1/4} = \left(\frac{\rho_s h\omega^2}{D}\right)^{1/4}$$

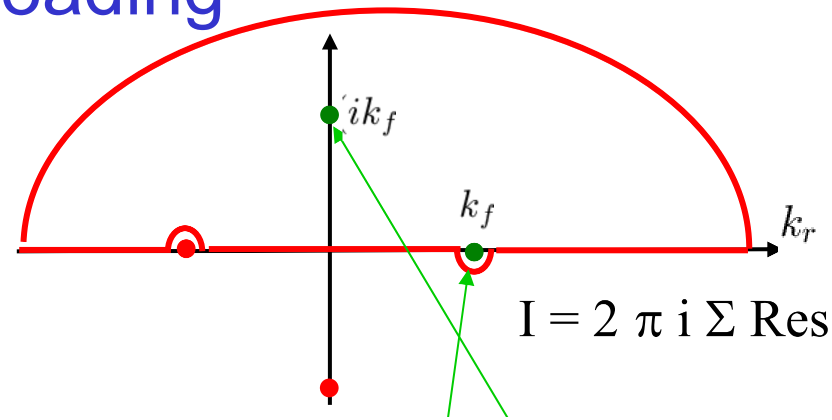
$$k_f = \frac{\omega}{\alpha}$$

Particle Velocity

$$\dot{w}_\omega(k_r) = \frac{-i\omega F_\omega}{2\pi D(k_r^4 - k_f^4)}$$

Point-driven Plate
Light Fluid Loading





Complex Contour Integration

$$\dot{w}_\omega(r) = \frac{F_\omega}{8\alpha^2 m_s}\left[H_0^{(1)}(k_f r) - H_0^{(1)}(ik_f r)\right]$$

$$= \frac{F_\omega}{8\alpha^2 m_s}\left[H_0^{(1)}(k_f r) - \frac{2i}{\pi}K_0(k_f r)\right]$$

$$H_0^{(1)}(k_r r) \rightarrow \sqrt{\frac{2}{\pi k_r r}}e^{i(k_r r - \pi/4)}$$

Inverse Hankel Transform

$$w(r) = \frac{F_\omega}{2\pi D}\int_0^\infty \frac{J_0(k_r r)}{k_r^4 - k_f^4}k_r dk_r$$

$$= \frac{F_\omega}{2\pi D}\int_{-\infty}^\infty \frac{H_0^{(1)}(k_r r)}{k_r^4 - k_f^4}k_r dk_r$$

$$J_0(x) = \frac{1}{2}\left(H_0^{(1)}(x) + H_0^{(2)}(x)\right)$$

$$= \frac{1}{2}\left(H_0^{(1)}(x) - H_0^{(1)}(-x)\right)$$

Drive-point Impedance

$$Z_p = F_\omega/\dot{w}(0) = 8\alpha^2 m_s$$

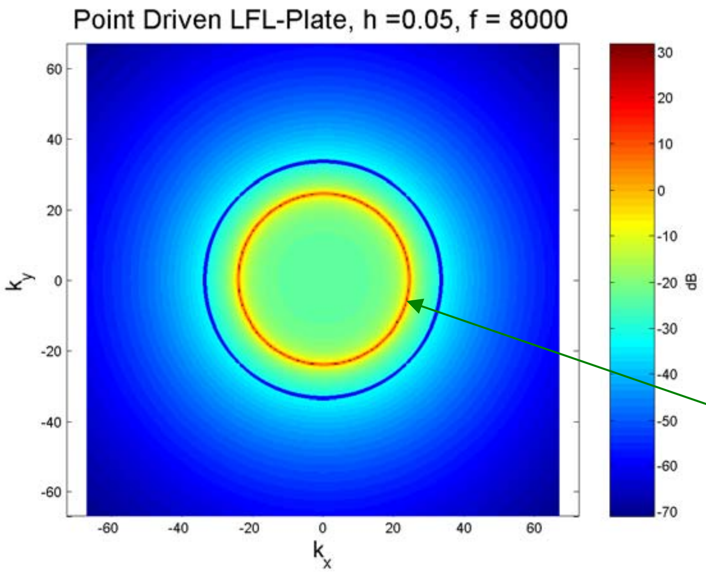
Flexural Wave Speed

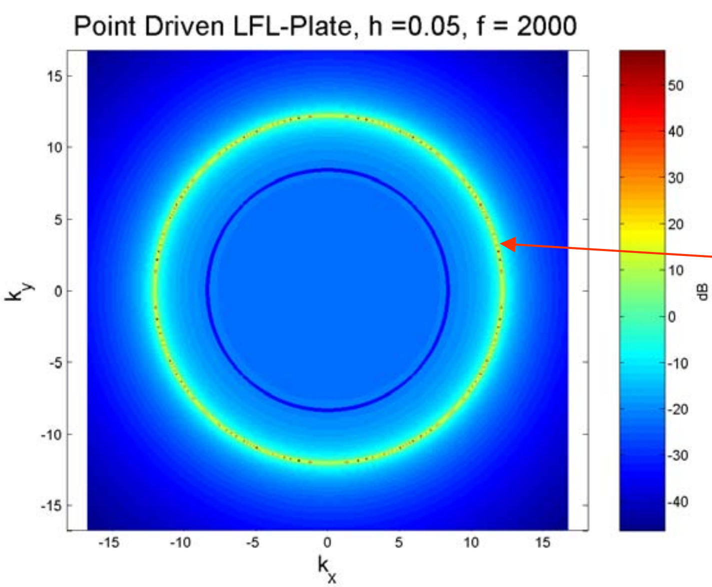
$$k_f = \omega/c_f$$

$$c_f = \omega/k_f = \alpha\sqrt{\omega}$$

Point Driven Plate

$$D(\theta,\phi) = -\frac{i\rho\omega}{2\pi}\dot{w}_\omega(k sin\theta)$$





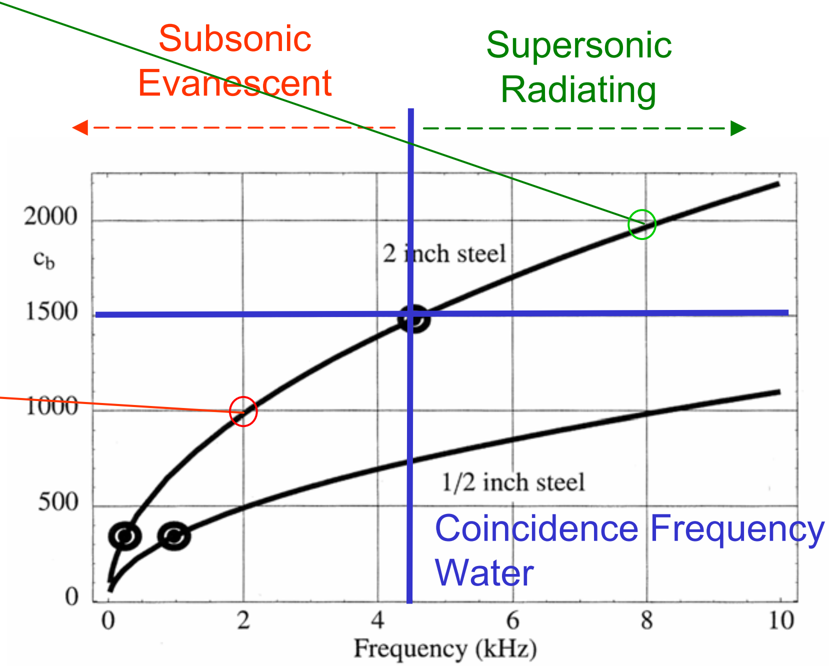
Radiated Pressure

$$p(R,\theta,\phi) = -\frac{i\rho\omega}{2\pi}\frac{e^{ikR}}{R}\dot{w}_\omega(k_r) = -\frac{i\rho\omega}{2\pi}\frac{e^{ikR}}{R}\dot{w}_\omega(k sin\theta)$$

$$\sin\theta_0 = k_f/k$$

Flexural Wave Speed

$$c_f \;=\; \omega/k_f = \alpha\sqrt{\omega}$$



Point-Driven Plate Radiation Exact Formulation

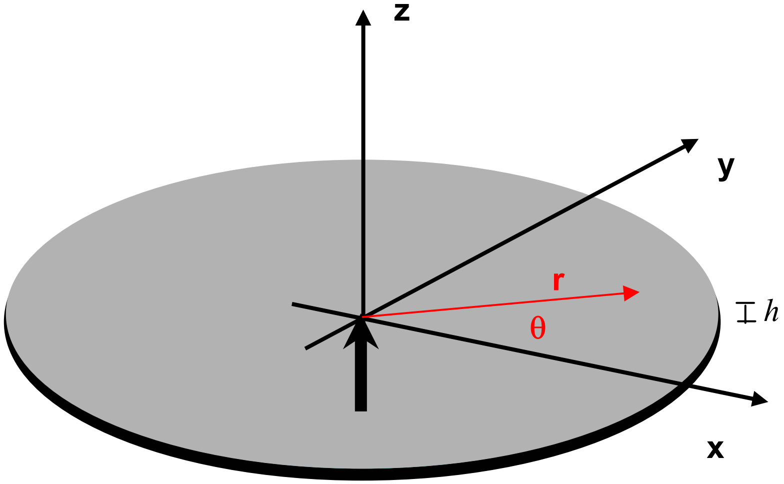


Plate Equation of Motion

$$D\left(\frac{d^2}{dr^2}+\frac{d}{rdr}\right)^2 w_\omega - \rho_s h\omega^2 w_\omega = F_\omega \frac{\delta(r)}{2\pi r} - p_a(r)$$

Fourier Transform

$$D\nabla^4 w_\omega(k_r) - \rho_s h \omega^2 w_\omega(k_r) = F_\omega - p_\omega(k_r)$$

$$p_\omega(k_r) = \frac{\rho\omega}{k_z}\dot{w}_\omega(k_r) = \frac{-i\rho\omega^2}{k_z} w_\omega(k_r)$$

$$D\nabla^4 w_\omega(k_r) - \left[\rho_s h + i\rho k_z^{-1}\right]\omega^2 w_\omega(k_r) = F_\omega$$

Vertical Plate Displacement

$$w_\omega(k_r) = \frac{F(\omega)}{2\pi D[k_r^4 - k_f^4(k_r)]}$$

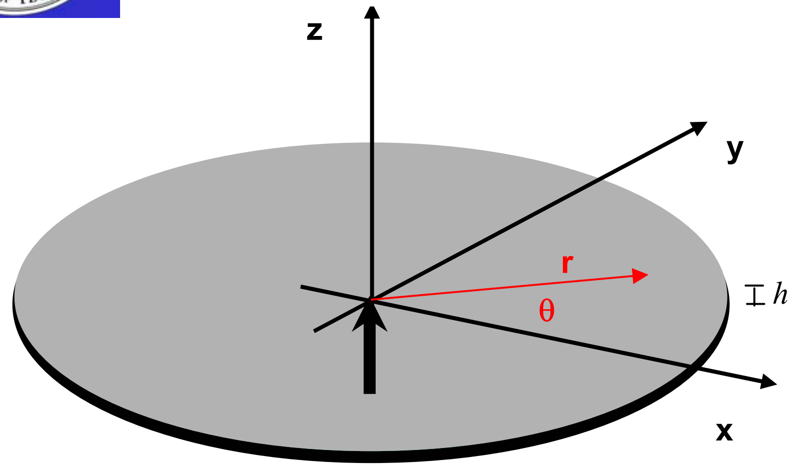
$k_r < k$: Radiation Damping
$k_r > k$: Added Mass

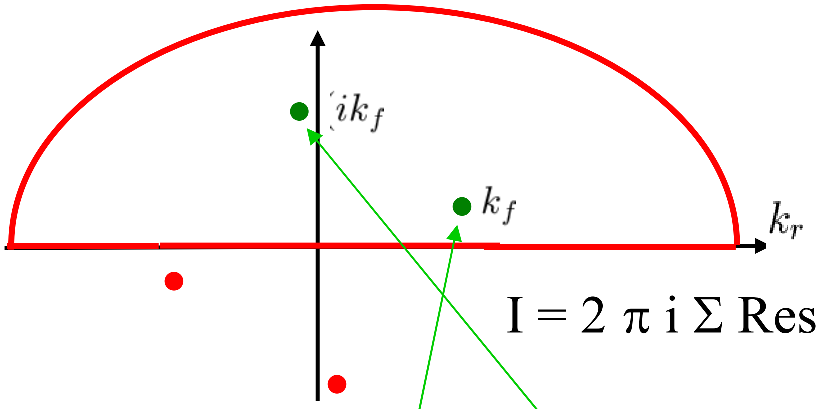
$$k_f(k_r) = \left(\frac{(m_s + i m_a(k_r))\omega^2}{D}\right)^{1/4} = \left(\frac{(\rho_s h + i\rho k_z^{-1})\omega^2}{D}\right)^{1/4}$$

Vertical Plate Velocity

$$\dot{w}_\omega(k_r) = \frac{-i\omega F_\omega}{2\pi D[k_r^4 - k_f^4(k_r)]}$$

Point-driven Fluid-loaded Plate
Exact Formulation





Complex Contour Integration

$$\dot{w}_\omega(r) = \frac{F_\omega}{8\alpha^2 m_s}\left[H_0^{(1)}(k_f r) - H_0^{(1)}(ik_f r)\right]$$

$$= \frac{F_\omega}{8\alpha^2 m_s}\left[H_0^{(1)}(k_f r) - \frac{2i}{\pi}K_0(k_f r)\right]$$

$$H_0^{(1)}(k_r r) \rightarrow \sqrt{\frac{2}{\pi k_r r}}e^{i(k_r r - \pi/4)}$$

Inverse Hankel Transform

$$\dot{w}_\omega(r) = \frac{-i\omega F_\omega}{2\pi D}\int_0^\infty \frac{J_0(k_r r)}{k_r^4 - k_f^4(k_r)}k_r dk_r$$

$$= \frac{-i\omega F_\omega}{2\pi D}\int_{-\infty}^\infty \frac{H_0^{(1)}(k_r r)}{k_r^4 - k_f^4(k_r)}k_r dk_r$$

Flexural Wavenumber Equation

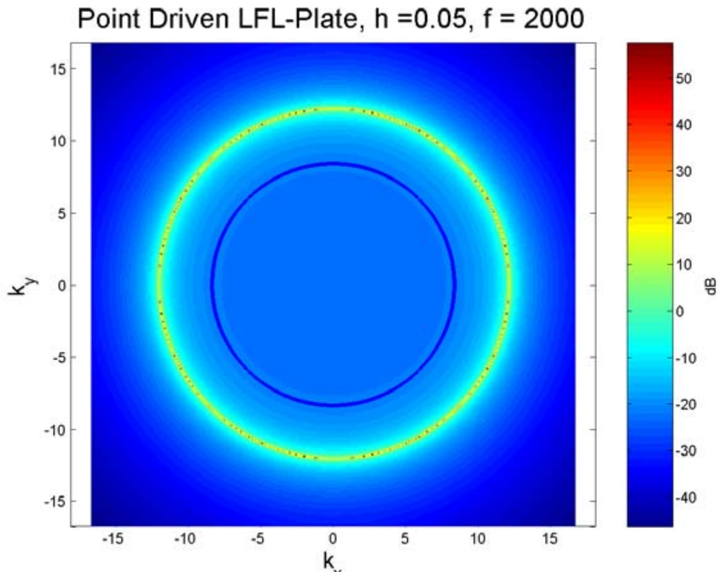
$$k_f = \left(\frac{(m_s + im_a(k_f))\omega^2}{D}\right)^{1/4} = \left(\frac{(\rho_s h + i\rho(k^2 - k_f^2)^{-1/2})\omega^2}{D}\right)^{1/4}$$

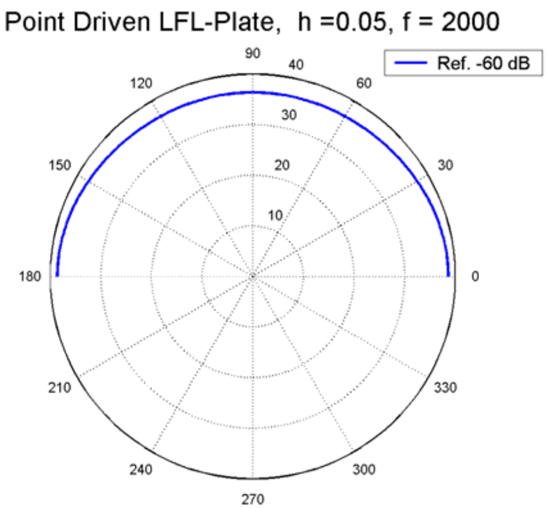
Directivity Function

$$D_\omega(\theta, \phi) = -\frac{\rho\omega}{2\pi}\dot{w}_\omega(k_r) = \frac{-\rho\omega^2 F_\omega}{4\pi^2 D[k_r^4 - k_f^4(k_r)]}$$

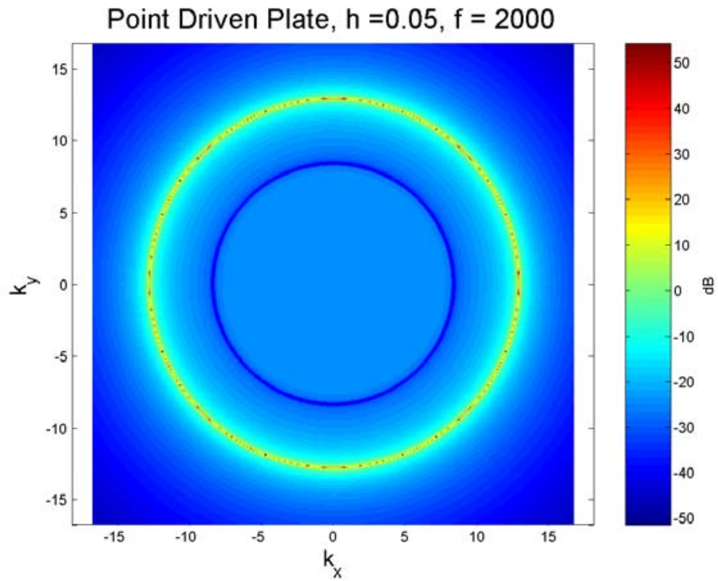
Point-driven Plate
Evanescent Frequency Regime
2 kHz

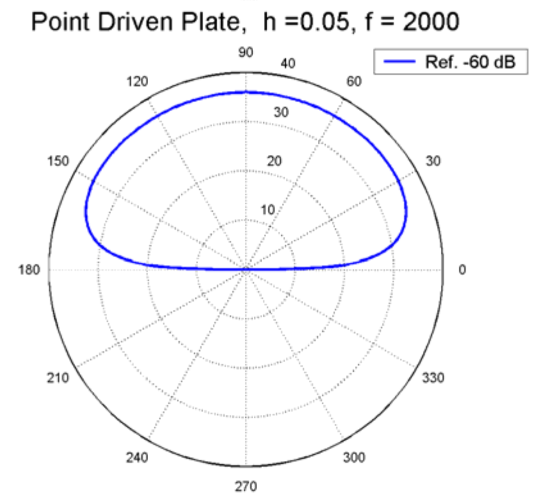
Light Fluid Loading





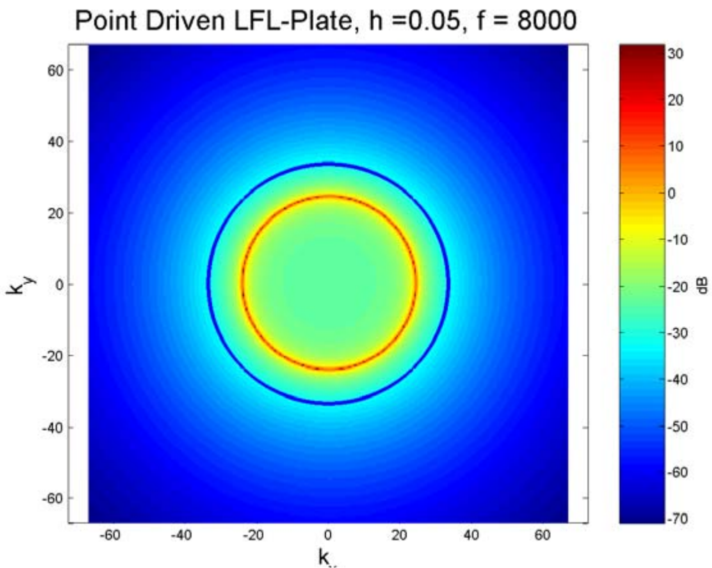
Exact

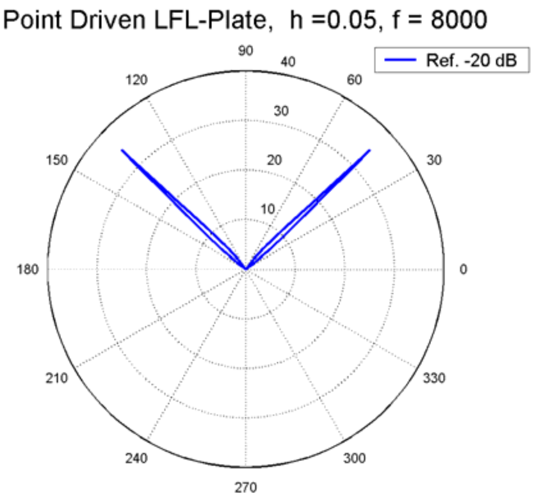




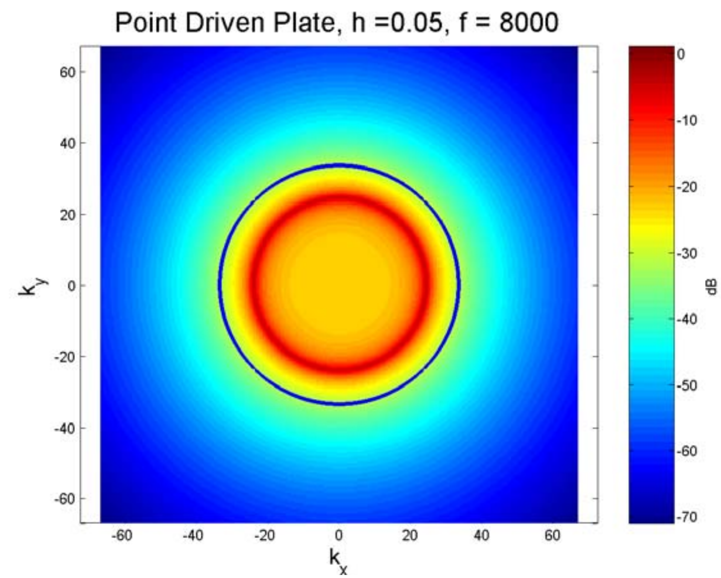
Point-driven Plate
Radiation Frequency Regime
8 kHz

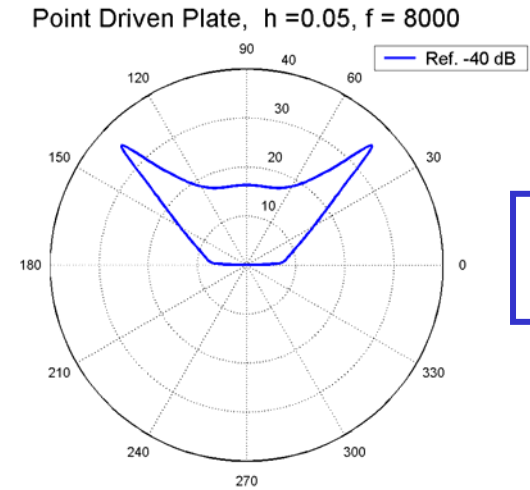
Light Fluid Loading





Exact



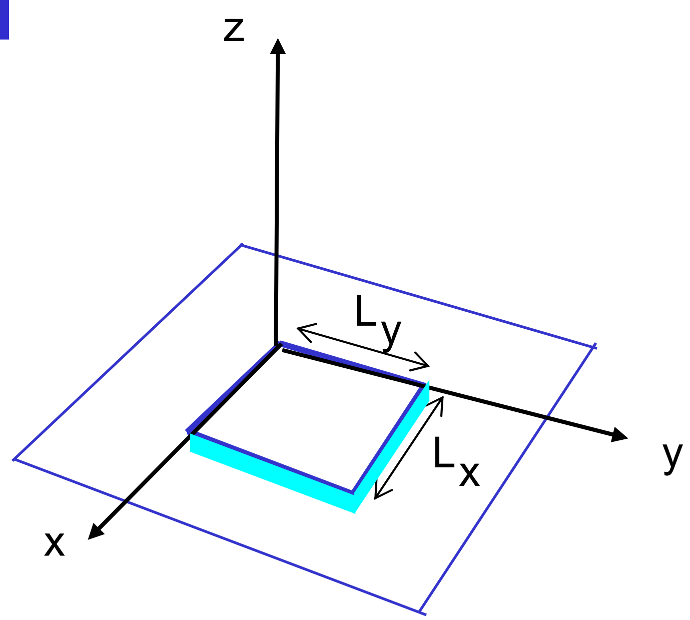


Radiation Damping

plate.m

```
%
% MATLAB script for plotting the directivity function for
% a pointdriven elastic plate
%
% Parameters:
% f     Frequency
% rho   Density
% c     Speed of Sound
% h     Plate thickness
% E     Young's modulus
% nu    Poisson's ratio
% rhos Plate density
clear
rhos=7700;
cp=5600;
h=0.05;
E=rhos*cp^2;
nu=0.33;
D= E *h^3/(12*(1-nu^2));
rho=1000;
c=1500;
f=8000;
omega=2*pi*f;
k=omega/c
ka=k;
figure(1);
hold off
kxm=2*ka;
nkx=300;
dkx=2*kxm/(nkx-1);
x=[-kxm:dkx:kxm];
y=x;
o=ones(1,nkx);
kx=x' * o;
ky=(y' *o)';
kr=abs(complex(kx,ky));
kf= ((rhos*h+i*rho./sqrt(k^2-complex(kr,0.0).^2))*omega^2/D).^0.25;
kfa=kf*a;
ss=-rho*omega^2./(D*(2*pi)^2 *(complex(kr,0).^4 -kfa.^4));
wavei(dba(ss)',x,y)
shading('flat')
axis('equal')
b=xlabel('k_x')
set(b,'FontSize',16);
b=ylabel('k_y')
set(b,'FontSize',16);
tit=['Point Driven Plate, h =' num2str(h) ', f = ' num2str(f)]
b=title(tit);
set(b,'FontSize',20);
nphi=361;
dphi=2*pi/(nphi-1);
phi=[0:dphi:2*pi];
xx=k*a*cos(phi);
yy=k*a*sin(phi);
hold on
b=plot(xx,yy,'b');
figure(2)
nphi=361.
dphi=2*pi/(nphi-1)
nth=181;
dth=0.5*pi/(nth-0.5);

phi=[0:dphi:(nphi-1)*dphi]' * ones(1,nth);
th=([dth/2:dth:pi/2]'*ones(1,nphi))';
kx=ka*sin(th).*cos(phi);
ky=ka*sin(th).*sin(phi);
kr=ka*sin(th);
kf= ((rhos*h+i*rho./sqrt(k^2-complex(kr,0.0).^2))*omega^2/D).^0.25;
kfa=kf*a;

ss=-rho*omega^2./(D*(2*pi)^2 *(complex(kr,0).^4 -kfa.^4));

ss=dba(ss);
sm=10.0*(ceil(0.1*max((max(ss))')));
for i=1:size(ss,1)
     for j=1:size(ss,2)
     ss(i,j)=max(ss(i,j),sm-40.0)-(sm-40.0);
end
end

xx=ss.*sin(th).*cos(phi);
yy=ss.*sin(th).*sin(phi);
zz=ss.*cos(th);

surfl(xx,yy,zz);
colormap('copper');
shading('flat');
axis('equal');
tit=['Point Driven Plate,  h =' num2str(h) ', f = '
      num2str(omega/(2*pi)) ]


b=title(tit);
set(b,'FontSize',20);

figure(3)
b=polar([pi/2-fliplr(th(1,:)) pi/2+th((nphi-1)/2+1,:)],
        [fliplr(ss(1,:)) ss((nphi-1)/2+1,:)]);
set(b,'LineWidth',2)
     b=legend(['Ref. ' num2str(sm-40.0) ' dB']);
set(b,'FontSize',14);

tit=['Point Driven Plate,  h =' num2str(h) ', f = '
      num2str(omega/(2*pi)) ]
b=title(tit);
set(b,'FontSize',20);
```

Simply-supported Elastic Plate



Homogeneous Equation of Motion

$$D\left[\frac{\partial^4 w_\omega}{\partial x^4} + 2\frac{\partial^4 w_\omega}{\partial x^2 \partial y^2} + \frac{\partial^4 w_\omega}{\partial y^4}\right] - \rho_s h \omega^2 w_\omega = 0$$

$$\nabla^4 w_\omega(x,y) - k_f^4 w_\omega(x,y) = 0$$

$$k_f = (\rho_s h \omega^2 / D)^{1/4} = \frac{\omega}{\alpha}$$

Moments

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right)$$

Boundary Conditions

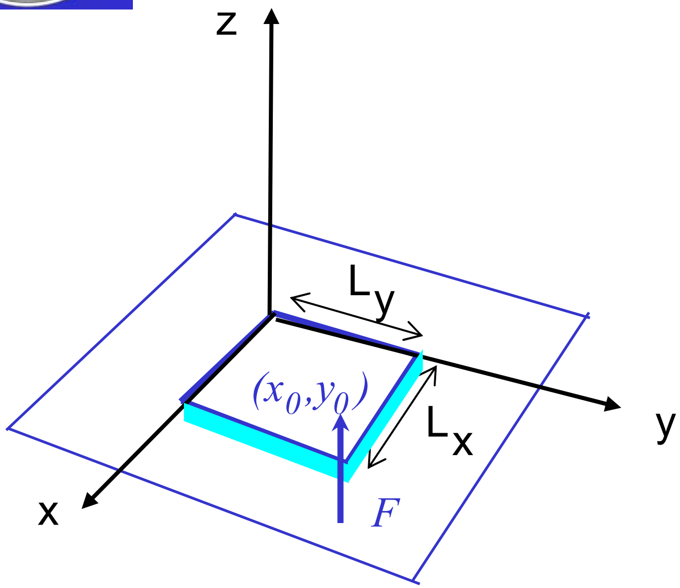
$$w(x,y), M_x(x,y) = 0, \quad x = 0, L_x$$

$$w(x,y), M_y(x,y) = 0, \quad y = 0, L_y$$

$$w_\omega(x,y), \frac{\partial^2 w_\omega(x,y)}{\partial x^2} \equiv 0, \quad x = 0, L_x$$

$$w_\omega(x,y), \frac{\partial^2 w_\omega(x,y)}{\partial y^2} \equiv 0, \quad y = 0, L_y$$

Point-driven Rectangular Elastic Plate



Homogeneous Equation of Motion

$$\nabla^4 w_\omega(x,y) - k_f^4 w_\omega(x,y) = 0$$

$$k_f = (\rho_s h \omega^2/D)^{1/4} = \frac{\omega}{\alpha}$$

Normal Modes

$$\Phi_{mn}(x,y) = \frac{2}{\sqrt{L_x L_y}} \sin(m\pi x/L_x)\sin(n\pi y/L_y), \quad m,n = 1,2,3\cdots$$

Solutions for

$$(m\pi/L_x)^2 + (n\pi/L_y)^2 = k_f^2$$

Orthogonality Relation

$$\int_0^{L_x}\int_0^{L_y} \Phi_{mn}(x,y)\Phi_{pq}(x,y)dxdy = \delta_{mp}\delta_{nq}$$

Dispersion Relation

$$\omega_{mn} = \alpha^2[((m\pi/L_x)^2 + (n\pi/L_y)^2]$$

Light Fluid Loading

$$D\left[\nabla^4 w_\omega(x,y) - k_f^4 w_\omega(x,y)\right] = F_\omega \delta(x-x_0)\delta(y-y_0) - p_\omega(x,y,0)$$

$$w_\omega(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}(\omega)\Phi_{mn}(x,y)$$

$$D\nabla^4\Phi_{mn}(x,y) = \rho_s h \omega_{mn}\Phi_{mn}(x,y)$$

$$\rho_s h(\omega_{mn}^2 - \omega^2)A_{mn} = \int_0^{L_x}\int_0^{L_y}\Phi_{mn}(x,y)F_\omega\delta(x-x_0)\delta(y-y_0)dxdy = F_\omega\Phi_{mn}(x_0,y_0)$$

Normal Mode Solution

$$w_\omega(x,y) = -\frac{F_\omega}{\rho_s h}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\Phi_{mn}(x_0,y_0)\Phi_{mn}(x,y)}{\omega^2 - \omega_{mn}^2}$$

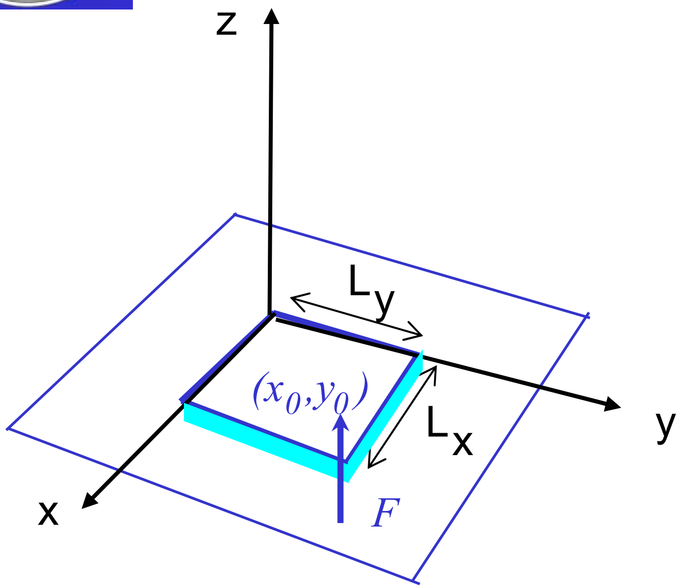
Transfer Mobility

$$Y_\omega = \frac{\dot{w}(x,y)}{F_\omega} = \frac{i\omega}{\rho_s h}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\Phi_{mn}(x_0,y_0)\Phi_{mn}(x,y)}{\omega^2 - \omega_{mn}^2}$$

Normal Modes of Simply-supported Elastic Plate

Image removed due to copyright considerations.
See Figure 2.26 in Williams, E. G. *Fourier Acoustics*.
London: Academic Press, 1999

Radiation from Point-driven Elastic Plate



$$\begin{aligned}
\Phi_{mn}(x,y) &= \frac{2}{\sqrt{L_x L_y}} \sin(k_{xm}x)\sin(k_{yn}y), \quad m,n = 1,2,3\cdots \\
k_{xm} &= m\pi/L_x \\
k_{ym} &= n\pi/L_y \\
k_{xm}^2 + k_{yn}^2 &= k_f^2
\end{aligned}$$

$$\dot{w}_\omega(x,y) = \frac{i\omega\Pi((x-L_x/2)/L_x)\Pi((y-L_y/2)/L_y)}{2\sqrt{L_x L_y}} \times \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}(\omega)(e^{ik_{xm}x} - e^{-ik_{xm}x})(e^{ik_{yn}y} - e^{-ik_{yn}y})$$

$$\dot{w}_\omega(k_x,k_y) = \frac{i\omega\sqrt{L_x L_y}}{2} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}(\omega)\left[\pm e^{-i(k_x \mp k_{xm})L_x/2}\mathrm{sinc}(\frac{(k_x \mp k_{xm})L_x}{2})\right] \times \left[\pm e^{-i(k_y \mp k_{yn})L_y/2}\mathrm{sinc}(\frac{(k_y \mp k_{yn})L_y}{2})\right]$$

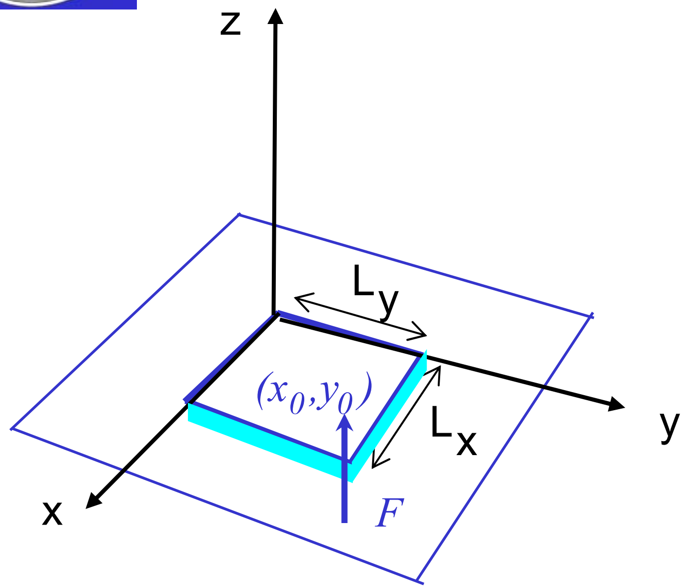
Fourier Transforms

$$\begin{aligned}
&\int_{-\infty}^{\infty} \Pi((x-L_x/2)/L_x)e^{\pm ik_{xm}x}e^{-ik_x x}dx \\
&= \left[\int_{-\infty}^{\infty} \Pi((x-L_x/2)/L_x)e^{-i(k_x x)}dx\right] * \delta(k_x \mp k_{xm}) \\
&= \left[e^{-ik_x L_x/2} L\mathrm{sinc}(k_x L_x/2)\right] * \delta(k_x \mp k_{xm}) \\
&= e^{-i(k_x \mp k_{xm})L/2} L\mathrm{sinc}((k_x \mp k_{xm})L/2)
\end{aligned}$$

Directivity Function

$$\begin{aligned}
D_\omega(\theta,\phi) &= -\frac{i\rho\omega}{2\pi}\dot{w}(k_x,k_y) \\
&= \frac{\rho\omega^2\sqrt{L_x L_y}}{4\pi} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}(\omega)\left[\pm e^{-i(k_x \mp k_{xm})L_x/2}\mathrm{sinc}(\frac{(k_x \mp k_{xm})L_x}{2})\right] \\
&\quad \times \left[\pm e^{-i(k_y \mp k_{yn})L_y/2}\mathrm{sinc}(\frac{(k_y \mp k_{yn})L_y}{2})\right]
\end{aligned}$$

Radiation Efficiency



Radiation Efficiency

$$S \equiv \frac{\Pi}{\Pi_0} \equiv \frac{\Pi}{\frac{1}{2}\rho c L_x L_y < |\dot{w}|^2 >}$$

RMS Velocity

$$< |\dot{w}|^2 > \equiv \frac{1}{L_x L_y} \int_0^{L_x} \int_0^{L_y} |\dot{w}(x,y)|^2 dx dy$$

Mode m,n

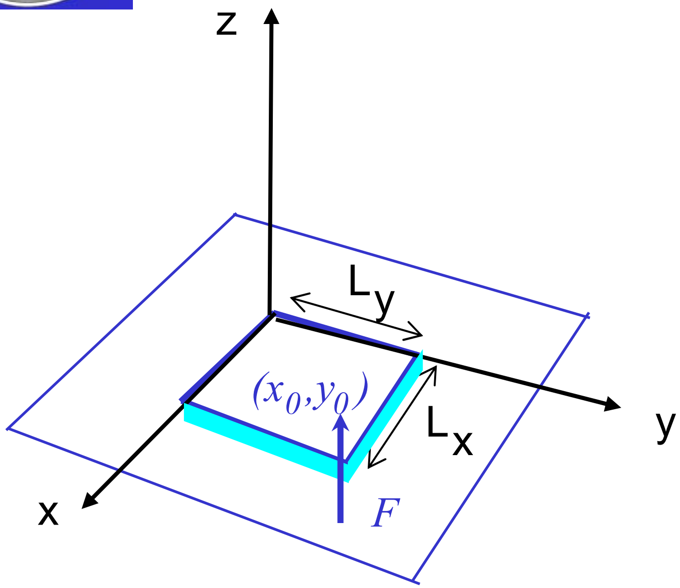
$$S_{mn} = \frac{\Pi}{\frac{1}{2}\rho c L_x L_y < |\dot{w}_{mn}|^2 >}$$

Low-order Modes – Square Plate

Image removed due to copyright considerations.
See Figure 2.27 in [Williams].

k/k_f

Radiation Efficiency



Radiation Efficiency

$$S \equiv \frac{\Pi}{\Pi_0} \equiv \frac{\Pi}{\frac{1}{2}\rho c L_x L_y < |\dot{w}|^2 >}$$

RMS Velocity

$$< |\dot{w}|^2 > \equiv \frac{1}{L_x L_y} \int_0^{L_x} \int_0^{L_y} |\dot{w}(x,y)|^2 dx dy$$

Mode m,n

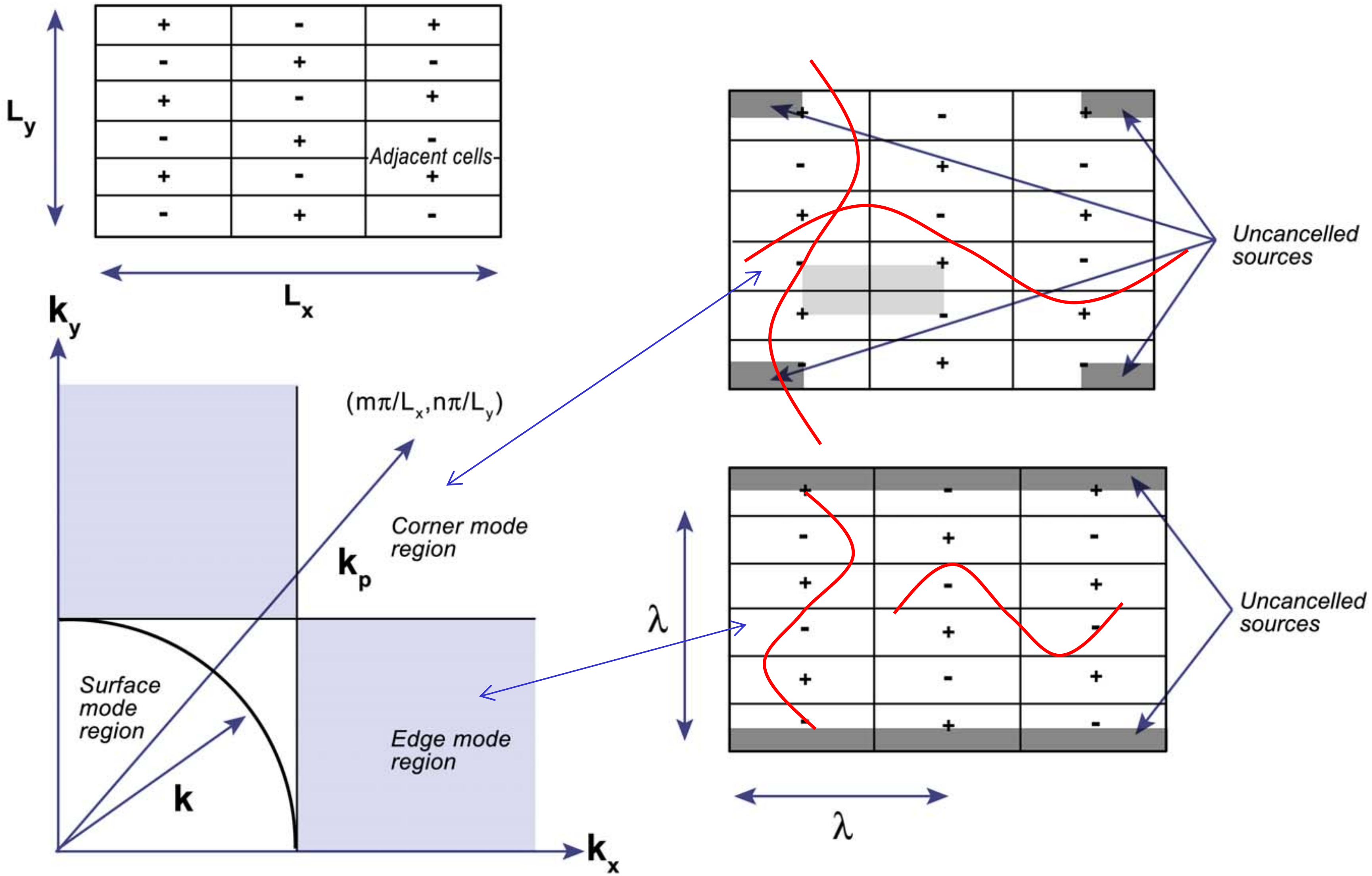
$$S_{mn} = \frac{\Pi}{\frac{1}{2}\rho c L_x L_y < |\dot{w}_{mn}|^2 >}$$

High-order Modes – Square Plate

Image removed due to copyright considerations.
See Figure 2.28 in [Williams].

k/k_f

Rectangular Elastic Plate Radiation Mode Types



Rectangular Elastic Plate Radiation Mode Excitation

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See Figure 2.34 in [Williams].

Rectangular Elastic Plate Radiation
Supersonic Intensity

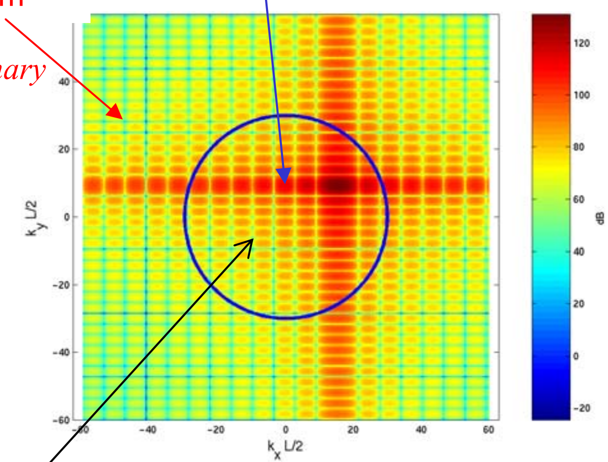
M,n = 11,9 – kL/2 = 1 – k/k_f =0.27

Radiating Spectrum

k_z *Real*

Evanescent Spectrum

k_z *Imaginary*



Supersonic Intensity

$$\Pi_\omega^{(s)}(x,y) = \frac{1}{2} Re[p_\omega^{(s)}(x,y)\dot{w}_\omega^{(s)}(x,y)*]$$

Image removed due to copyright considerations.
See Figure 2.37 in [Williams].

Supersonic Intensity

Image removed due to copyright considerations.
See Figure 2.38 in [Williams].

Normal Acoustic Intensity

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See Figure 2.39 in [Williams].