Computational Ocean Acoustics

- Ray Tracing
- Wavenumber Integration
- Normal Modes
- Parabolic Equation

Normal Modes

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Normal Modes

Mathematical Derivation

Point Source in Cylindrical Geometry

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)+\rho(z)\frac{\partial}{\partial z}\left(\frac{1}{\rho(z)}\frac{\partial p}{\partial z}\right)+\frac{\omega^2}{c^2(z)}p=-\frac{\delta(r)\,\delta(z-z_s)}{2\pi r}\,.$$

Separation of variables

Substitute $p(r,z)=\Phi(r)\Psi(z)$ and divide by $\Phi(r)\Psi(z)$,

$$\frac{1}{\Phi}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Phi}{dr}\right)\right]+\frac{1}{\Psi}\left[\rho(z)\frac{d}{dz}\left(\frac{1}{\rho(z)}\frac{d\Psi}{dz}\right)+\frac{\omega^2}{c^2(z)}\Psi\right]=0\,.$$

Each component equal to a separation constant k_{rm}^2,

$$\rho(z)\frac{d}{dz}\left[\frac{1}{\rho(z)}\frac{d\Psi_m(z)}{dz}\right]+\left[\frac{\omega^2}{c^2(z)}-k_{rm}^2\right]\Psi_m(z)=0\,,$$

Modal Equation

Boundary Conditions

$$\Psi(0)=0\,,\qquad \left.\frac{d\Psi}{dz}\right|_{z=D}=0\,.$$

Classical Sturm-Liouville Eigenvalue Problem

- Modal equation has infinite set of solutions – modes of vibrating string
- Modes characterized by
  - Mode shape Ψ (z) (eigenfunction)
  - Propagation constant. k_{rm}. k_{rm}^2 real (eigenvalue)
  - m-th mode has m zeros in [0,D]
  - $k_{rm} < \omega / c_{min}$
- Modes are Orthogonal
- Modes form a Complete Set

Modal Orthogonality

$$\int_0^D \frac{\Psi_m(z)\,\Psi_n(z)}{\rho(z)}\,dz = 0\,, \qquad \text{for} \qquad m \neq n\,.$$

Mode Normalization

$$\int_0^D \frac{\Psi_m^2(z)}{\rho(z)}\,dz = 1\,.$$

Complete Mode Set

$$p(r,z) = \sum_{m=1}^{\infty} \Phi_m(r)\,\Psi_m(z)\,.$$

Complete Mode Set

$$p(r,z) = \sum_{m=1}^{\infty} \Phi_m(r)\, \Psi_m(z)\ .$$

Substitution into Helmholtz Equation

$$\sum_{m=1}^{\infty} \left\{ \frac{1}{r}\frac{d}{dr}\left(r\,\frac{d\Phi_m(r)}{dr}\right)\Psi_m(z) + \boxed{\Phi_m(r)\left[\rho(z)\,\frac{d}{dz}\left(\frac{1}{\rho(z)}\,\frac{d\Psi_m(z)}{dz}\right) + \frac{\omega^2}{c^2(z)}\,\Psi_m(z)\right]} \right\} = -\frac{\delta(r)\,\delta(z-z_s)}{2\pi r}\ .$$

From Mode Equation

$$\sum_{m=1}^{\infty} \left\{ \frac{1}{r}\frac{d}{dr}\left(r\,\frac{d\Phi_m(r)}{dr}\right)\Psi_m(z) + \boxed{k_{rm}^2\,\Phi_m(r)\,\Psi_m(z)} \right\} = -\frac{\delta(r)\,\delta(z-z_s)}{2\pi r}.$$

Apply the operator,

$$\int_0^D (\cdot)\,\frac{\Psi_n(z)}{\rho(z)}\,dz\ ,$$

Othogonality yields

$$\frac{1}{r}\frac{d}{dr}\left[r\,\frac{d\Phi_n(r)}{dr}\right] + k_{rn}^2\,\Phi_n(r) = -\frac{\delta(r)\,\Psi_n(z_s)}{2\pi r\,\rho(z_s)}\ .$$

Solution

$$\boxed{\Phi_n(r) = \frac{i}{4\,\rho(z_s)}\,\Psi_n(z_s)\,H_0^{(1,2)}(k_{rn}r)\ .}$$

Modal Field Solution

$$\boxed{p(r,z) = \frac{i}{4\,\rho(z_s)} \sum_{m=1}^{\infty} \Psi_m(z_s)\,\Psi_m(z)\,H_0^{(1)}(k_{rm} r)\,,}$$

Asymptotic Hankel function

$$\boxed{p(r,z) \simeq \frac{i}{\rho(z_s)\sqrt{8\pi r}}\, e^{-i\pi/4} \sum_{m=1}^{\infty} \Psi_m(z_s)\,\Psi_m(z)\,\frac{e^{ik_{rm} r}}{\sqrt{k_{rm}}}\,.}$$

Transmission Loss

$$\mathrm{TL}(r,z) = -20\log\left|\frac{p(r,z)}{p_0(r=1)}\right|\,,$$

where $p_0(r)$ is the *free space* field

$$p_0(r) = \frac{e^{ik_0 r}}{4\pi r}\,,$$

$$\boxed{\mathrm{TL}(r,z) \simeq -20\log\left|\frac{1}{\rho(z_s)}\sqrt{\frac{2\pi}{r}} \sum_{m=1}^{\infty} \Psi_m(z_s)\,\Psi_m(z)\,\frac{e^{ik_{rm} r}}{\sqrt{k_{rm}}}\right|\,.}$$

Incoherent Transmission Loss

$$\mathrm{TL}_{\mathrm{Inc}}(r,z) \simeq -20\log\frac{1}{\rho(z_s)}\sqrt{\frac{2\pi}{r}}\sqrt{\sum_{m=1}^{\infty}\left|\Psi_m(z_s)\,\Psi_m(z)\,\frac{e^{ik_{rm} r}}{\sqrt{k_{rm}}}\right|^2}\,.$$

Line Source in Plane Geometry

Cartesian Helmholts Equation

$$\frac{\partial^2 p}{\partial x^2} + \rho(z)\,\frac{\partial}{\partial z}\left(\frac{1}{\rho(z)}\,\frac{\partial p}{\partial z}\right) + \frac{\omega^2}{c^2(z)}\,p = -\delta(x)\,\delta(z - z_s)\,.$$

Solution of form

$$p(x,z) = \sum_{m=1}^{\infty} \Phi_m(x)\,\Psi_m(z)\,,$$

Substitution

$$\sum_{m=1}^{\infty} \left\{ \frac{d^2\Phi_m(x)}{dx^2}\,\Psi_m(z) + \boxed{\Phi_m(x)\left[\rho(z)\,\frac{d}{dz}\left(\frac{1}{\rho(z)}\,\frac{d\Psi(z)}{dz}\right) + \frac{\omega^2}{c^2(z)}\,\Psi_m(z)\right]} \right\} = -\delta(x)\,\delta(z - z_s)\,.$$

Same mode equation as for cylindrical coordinates

$$\sum_{m=1}^{\infty} \left[\frac{d^2\Phi_m(x)}{dx^2}\,\Psi_m(z) + \boxed{k_{xm}^2\,\Phi_m(x)\,\Psi_m(z)} \right] = -\delta(x)\,\delta(z - z_s)\,.$$

Apply operator

$$\int_0^D (\cdot)\, \frac{\Psi_n(z)}{\rho(z)}\, dz\,,$$

Orthogonality yields

$$\frac{d^2\Phi_n(x)}{dx^2} + k_{xn}^2\, \Phi_n(x) = \frac{-\delta(x)\, \Psi_n(z_s)}{\rho(z_s)}\,.$$

Range Solution

$$\Phi_n(x) = \frac{i}{2\,\rho(z_s)}\, \Psi_n(z_s)\, \frac{e^{\pm i k_{xn} x}}{k_{xn}}\,.$$

Modal Solution in Plane Geonmetry

$$p(x,z) = \frac{i}{2\,\rho(z_s)} \sum_{m=1}^{\infty} \Psi_m(z_s)\, \Psi_m(z)\, \frac{e^{i k_{xm} |x|}}{k_{xm}}.$$

Transmission Loss

Free Space Solution

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial p_0}{\partial r}\right) + \frac{\omega^2}{c^2(z)}\,p_0 = -\frac{\delta(r)}{r}\,,$$

$$p_0(r) = \frac{i}{4}\,H_0^{(1)}(k_0 r)\,,$$

Transmission Loss

$$\frac{p(x,z)}{p_0(r)|_{r=1}} = \frac{2}{\rho(z_s)\,H_0^{(1)}(k_0)} \sum_{m=1}^{\infty} \Psi_m(z_s)\,\Psi_m(z)\,\frac{e^{ik_{xm}|x|}}{k_{xm}}\,.$$

Asymptotic Normalization

$$\boxed{\frac{p(x,z)}{p_0(r=1)} \simeq \frac{\sqrt{2\pi k_0}}{\rho(z_s)}\,e^{-i(k_0-\pi/4)} \sum_{m=1}^{\infty} \Psi_m(z_s)\,\Psi_m(z)\,\frac{e^{ik_{xm}|x|}}{k_{xm}}\,.}$$

$$\mathrm{TL}(x,z) = -20\log\left|\frac{p(x,z)}{p_0(r=1)}\right|\,.$$

Modal Expansion of the Green's Function

Depth-separated Helmholtz Equation

$$\rho(z)\,\frac{d}{dz}\left[\frac{1}{\rho(z)}\,\frac{d\,g(z)}{dz}\right] + \left[\frac{\omega^2}{c^2(z)} - k_r^2\right] g(z) = -\frac{\delta(z-z_s)}{2\pi}\,.$$

Modal Expansion of Delta Function

$$\delta(z-z_s) = \sum_m a_m \Psi_m(z)\,.$$

Apply operator

$$\int_0^D (\cdot)\,\frac{\Psi_n(z)}{\rho(z)}\,dz\,,$$

Orthogonality yields

$$a_n = \frac{\Psi_n(z_s)}{\rho(z_s)}\,.$$

Modal Solution for depth-separated Helmholtz Equation

$$g(z) = \sum_m b_m \Psi_m(z) \, .$$

$$\sum_{m=1}^{\infty} b_m \left[\rho(z) \frac{d}{dz} \left(\frac{1}{\rho(z)} \frac{d\Psi_m(z)}{dz} \right) + \left(\frac{\omega^2}{c^2(z)} - k_r^2 \right) \Psi_m(z) \right]$$
$$= -\frac{1}{2\pi} \sum_m \frac{\Psi_n(z_s)}{\rho(z_s)} \Psi_m(z) \, .$$

Rewrite as

$$\sum_{m=1}^{\infty} b_m \left(\boxed{k_{rm}^2} - k_r^2 \right) \Psi_m(z) = -\frac{1}{2\pi} \sum_m \frac{\Psi_n(z_s)}{\rho(z_s)} \Psi_m(z) \, .$$

Orthogonality yields

$$\left(k_{rm}^2 - k_r^2 \right) b_n = -\frac{\Psi_n(z_s)}{2\pi\rho(z_s)} \, .$$

Solve for b_n and substitute back into modal solution to yield

$$g(z) = \frac{1}{2\pi\rho(z_s)} \sum_m \frac{\Psi_m(z_s)\, \Psi_m(z)}{k_r^2 - k_{rm}^2} \, .$$

Green's function has singularities at values of k_r corresponding to the modal wavenumbers k_{rm}.

The Isovelocity Problem

$$\Psi_m(z) = A\sin(k_z z) + B\cos(k_z z)\ ,$$

$$k_z = \sqrt{\left(\frac{\omega}{c}\right)^2 - k_r^2}\ .$$

Bottom Boundary Condition

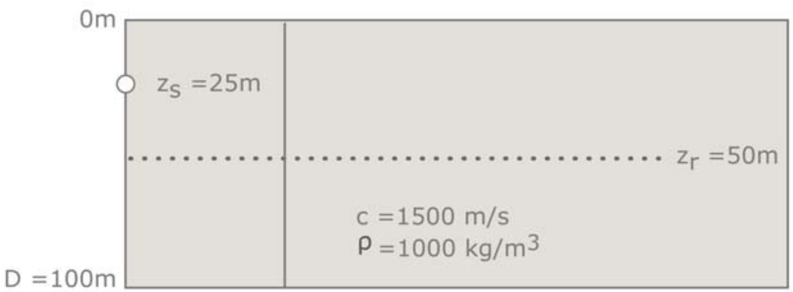
$$Ak_z\ \cos(k_z D) = 0\ ,$$

$$k_z D = \left(m - \frac{1}{2}\right)\pi\ , \qquad m = 1, 2, \ldots\ ,$$

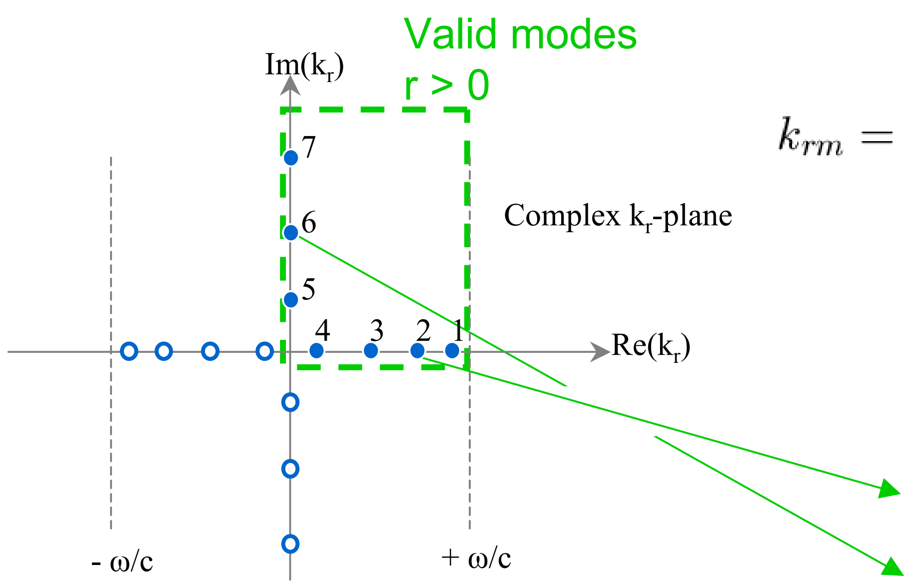
$$k_{rm} = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left[\left(m - \frac{1}{2}\right)\frac{\pi}{D}\right]^2}\ , \qquad m = 1, 2, \ldots$$

Eigenfunctions

$$\Psi_m(z) = \sqrt{\frac{2\rho}{D}}\ \sin(k_{zm} z)\ ,$$



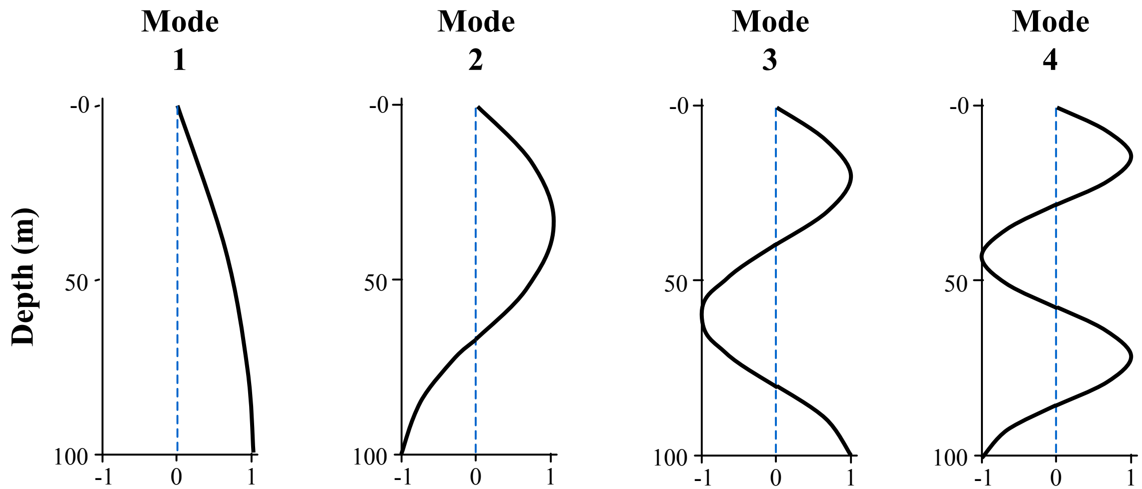
Schematic of the isovelocity problem.



Location of eigenvalues for the isovelocity problem.

k_{rm} real — Propagating Modes

k_{rm} imaginary — Evanescent Modes



Selected modes of the isovelocity problem.

Modal Cut-off Frequency

$$f_0 = \frac{c}{4D}\,.$$

Modal Exspansion

$$p(r,z) = \frac{i}{2D}\sum_{m=1}^{\infty} \sin(k_{zm}z_s)\sin(k_{zm}z)\,H_0^{(1)}(k_{rm}r)\,.$$

Intensity

$$I(r,z) = \left|\frac{1}{D}\sqrt{\frac{8\pi}{r}}\sum_{m=1}^{\infty}\sin(k_{zm}z_s)\sin(k_{zm}z)\,\frac{e^{ik_{rm}r}}{\sqrt{k_{rm}}}\right|^2\,.$$

Modal Interference

$$
\begin{aligned}
I(r,z) &= \frac{8\pi}{rD^2}\left|\sum_{m=1}^{\infty} A_m\, e^{ik_{rm}r}\right|^2 \\
&= \frac{8\pi}{rD^2}\left[\sum_m A_m^2 + \sum_m \sum_{n>m} 2A_m A_n \cos(\Delta k_{mn} r)\right] ,
\end{aligned}
$$

where

$$\Delta k_{mn} = k_{rm} - k_{rn} ,$$

and

$$A_m = \frac{\sin(k_{zm} z_s)\sin(k_{zm} z)}{\sqrt{k_{rm}}} .$$

[See Fig 5.4 in Jensen, Kuperman, Porter and Schmidt. *Computational Ocean Acoustics.* New York: Springer-Verlag, 2000.]

A Generalized Derivation

Pekeris Waveguide

Bottom Field

$$\Psi_b(z) = B e^{-\gamma_b z} + \cancel{C e^{\gamma_b z}} \,,$$

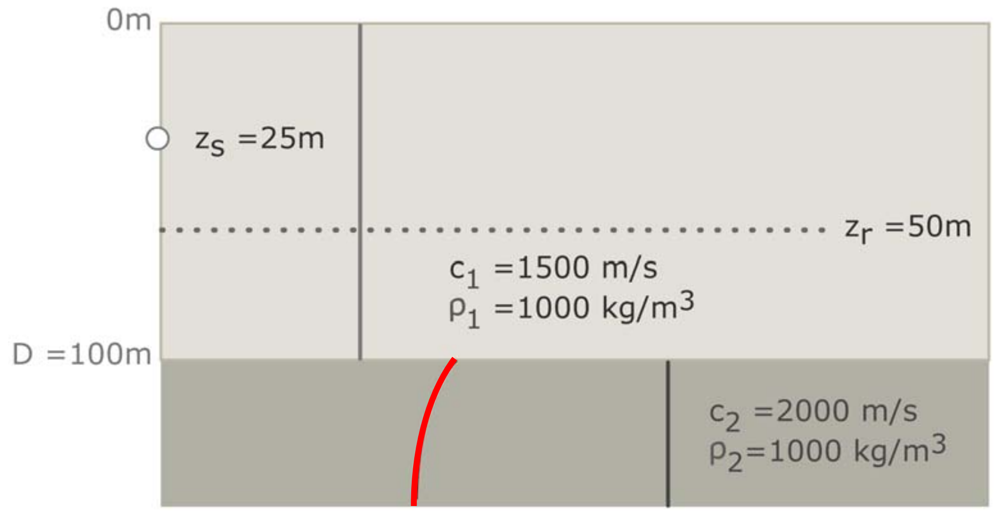
$$\gamma_b \equiv -i k_{z,b} = \sqrt{k_r^2 - \left(\frac{\omega}{c_b}\right)^2} \,,$$

Field at Seabed

$$\begin{aligned} \Psi(D) &= B\, e^{-\gamma_b D} \,, \\ \frac{d\Psi(D)/dz}{\rho} &= -B\, \frac{\gamma_b\, e^{-\gamma_b D}}{\rho_b} \,, \end{aligned}$$

Impedance Condition at Seabed

$$\frac{\rho\, \Psi(D)}{d\Psi(D)/dz} = -\frac{\rho_b}{\gamma_b(k_r^2)} \,.$$



Schematic of the Pekeris waveguide.

Mode Equation

$$\begin{aligned} \frac{d^2\Psi(z)}{dz^2} + \left[\frac{\omega^2}{c^2(z)} - k_r^2\right] \Psi(z) &= 0 \,, \\ \Psi(0) &= 0 \,, \\ f(k_r^2)\, \Psi(D) + \frac{g(k_r^2)}{\rho} \frac{d\Psi(D)}{dz} &= 0 \,. \end{aligned}$$

Seabed Impedance Condition yields

$$f(k_r^2) = 1 \,, \qquad g(k_r^2) = \rho_b \Big/ \sqrt{k_r^2 - \left(\frac{\omega}{c_b}\right)^2} \,.$$

Modal Equation
=
Homogeneous, Depth-Separated Helmholtz Equation (DSHE)

Solution

$$G(z, z_s; k_r) = -\frac{1}{2\pi} \frac{p_1(z_<; k_r)\, p_2(z_>; k_r)}{W(z_s; k_r)} ,$$

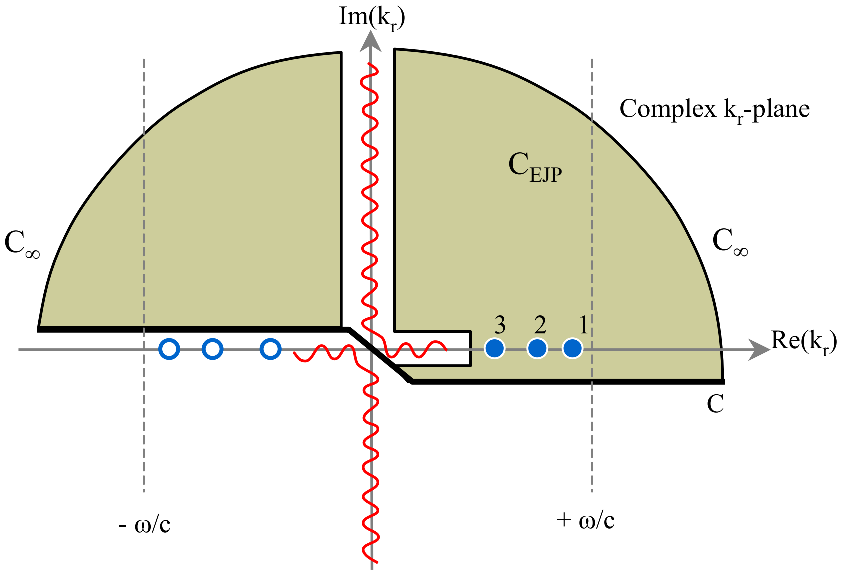
where $z_< = \min(z, z_s)$ and $z_> = \max(z, z_s)$.

Wronskian

$$W(z; k_r) = p_1(z; k_r)\, p_2'(z; k_r) - p_1'(z; k_r)\, p_2(z; k_r) ,$$

Operator Form of DSHE

$$\begin{aligned} \mathcal{L}(k_r)p_1 &= 0 , \qquad \mathcal{B}_1 p_1 = 0 , \\ \mathcal{L}(k_r)p_2 &= 0 , \qquad \mathcal{B}_2 p_2 = 0 . \end{aligned}$$



Location of eigenvalues for the Pekeris problem using the EJP branch cut.

Contour Integral

$$\int_{-\infty}^{\infty} + \int_{C_\infty} + \int_{C_{EJP}} = 2\pi i \sum_{m=1}^{M} \mathrm{res}(k_{rm}) \ ,$$

$\mathrm{res}(k_{rm})$: residue of the mth pole enclosed by the contour.

$$p(r,z) = \frac{i}{2} \sum_{m=1}^{M} \frac{p_1(z_<;k_{rm})\, p_2(z_>;k_{rm})}{\partial W(z_s;k_r)/\partial k_r|_{k_r=k_{rm}}} H_0^{(1)}(k_{rm}r)\, k_{rm} - \int_{C_{EJP}} ,$$

where k_{rm} is the mth zero of the Wronskian, ordered such that $\mathrm{Re}\{k_{r1}\} > \mathrm{Re}\{k_{r2}\} > \cdots$.

Branch Cut Selection

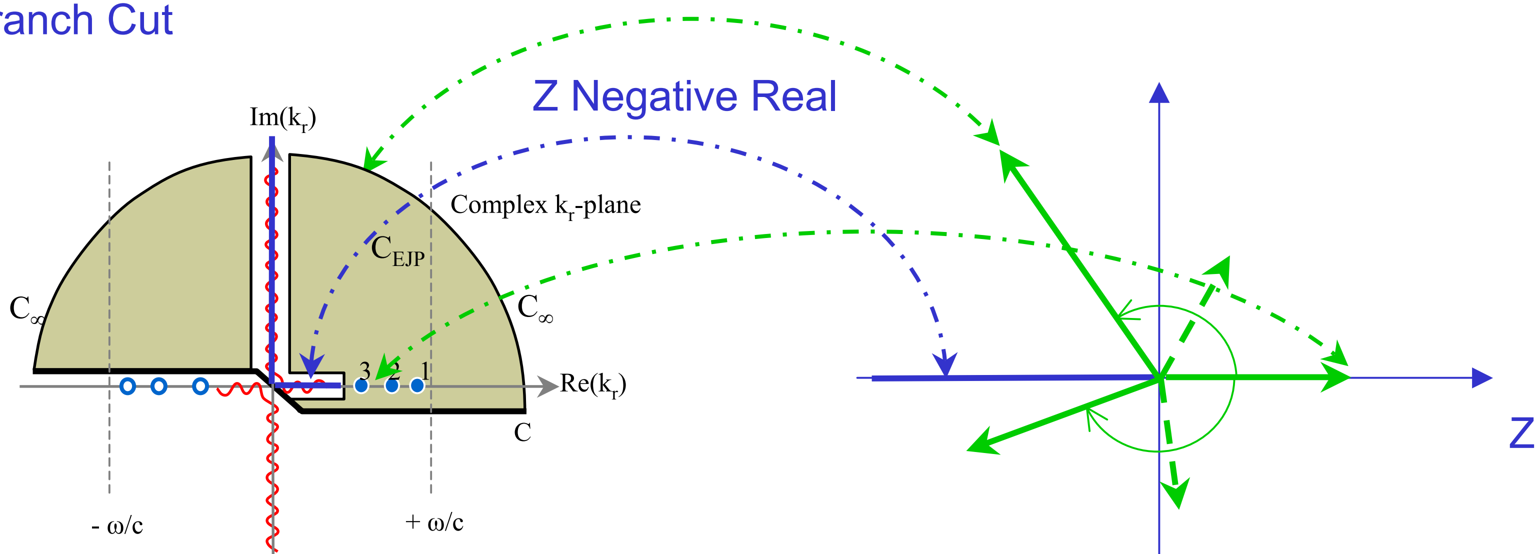
$$\gamma_b \equiv -ik_{z,b} = \sqrt{k_r^2 - \left(\frac{\omega}{c_b}\right)^2},$$

Complex Square Root

Z = R exp(iθ) = R exp(iθ+n2π)

$Z^{1/2}$= $R^{1/2}$exp(iθ/2+nπ)

EJP Branch Cut



$Re(\gamma_b) > 0$

$Re(Z^{1/2}) > 0$

EJP Brach Cut: Bottom field decaying for all k_r => Physical Riemann Sheet

Characteristic Equation

$$W(k_{rm}) = 0$$

$W(k_{rm}) = 0 \Rightarrow p_{1,2}(z;k_{rm})$ are linearly dependent.

Eigenfunctions

$$\Psi_m(z) = p_1(z;k_{rm}) = p_2(z;k_{rm})$$

which satisfies

$$\mathcal{L}(k_{rm})\Psi_m = 0, \qquad \mathcal{B}_1\Psi_m = \mathcal{B}_2\Psi_m = 0\,.$$

Modal Field Expansion

$$p(r,z) = \frac{i}{2}\sum_{m=1}^{M} \frac{\Psi_m(z_s)\,\Psi_m(z)}{\partial W(z_s;k_r)/\partial k_r|_{k_r=k_{rm}}} H_0^{(1)}(k_{rm}r)\,k_{rm} - \int_{C_{EJP}}\,.$$

Derivative of Wronskian

$$\left.\frac{\partial W/\partial k_r}{\rho(z_s)}\right|_{k_{rm}} = 2k_{rm}\int_0^D \frac{\Psi_m^2(z)}{\rho(z)}\,dz - \left.\frac{d(f/g)^T}{dk_r}\right|_{k_{rm}} \Psi_m^2(0) + \left.\frac{d(f/g)^B}{dk_r}\right|_{k_{rm}} \Psi_m^2(D).$$

Pressure Field

$$p(r,z) = \frac{i}{4\,\rho(z_s)} \sum_{m=1}^{M} \Psi_m(z_s)\,\Psi_m(z)\,H_0^{(1)}(k_{rm}r) - \int_{C_{EJP}},$$

Mode Normalization

$$\int_0^D \frac{\Psi_m^2(z)}{\rho(z)}\,dz - \frac{1}{2k_{rm}}\left.\frac{d(f/g)^T}{dk_r}\right|_{k_{rm}} \Psi_m^2(0) + \frac{1}{2k_{rm}}\left.\frac{d(f/g)^B}{dk_r}\right|_{k_{rm}} \Psi_m^2(D) = 1.$$

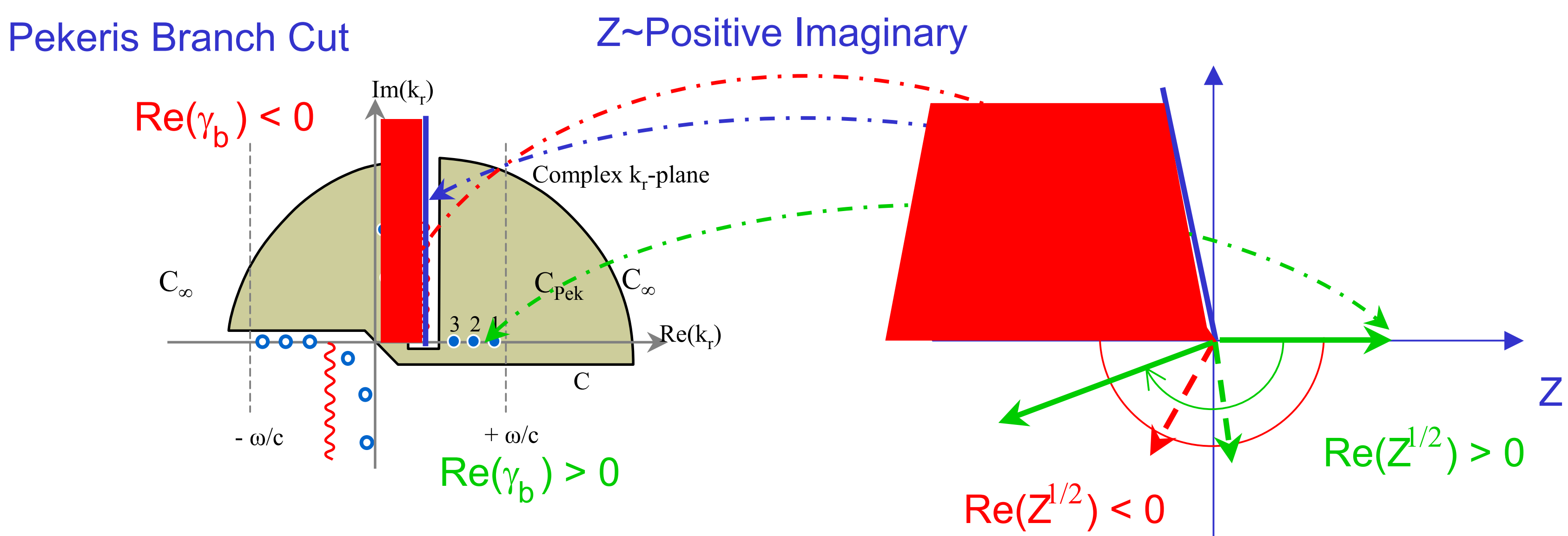
Branch Cut Selection

$$\gamma_b \equiv -ik_{z,b} = \sqrt{k_r^2 - \left(\frac{\omega}{c_b}\right)^2},$$

Complex Square Root

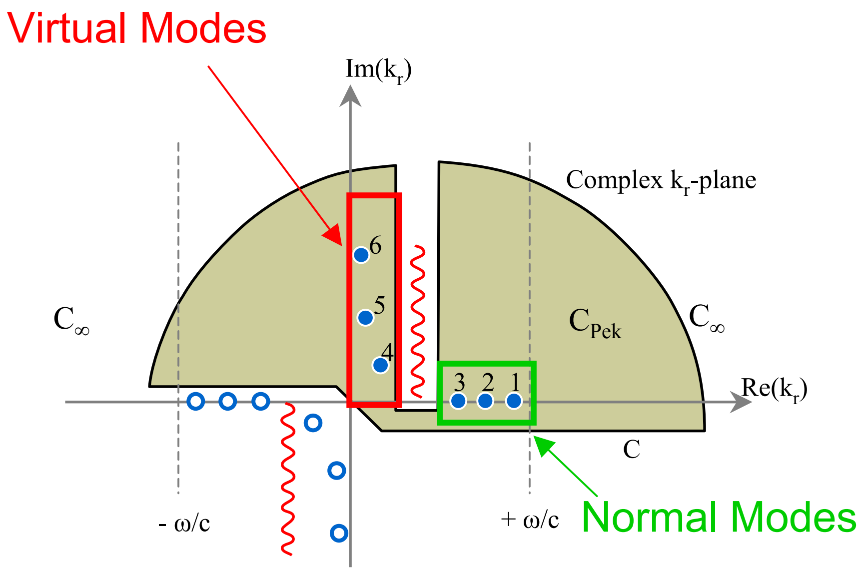
$$Z = R\exp(i\theta) = R\exp(i\theta + n2\pi)$$

$$Z^{1/2} = R^{1/2}\exp(i\theta/2 + n\pi)$$



Pekeris Branch Cut: Uncovers Virtual Modes on un-physical Riemann Sheet

Pekeris Branch Cut



Location of eigenvalues for the Pekeris problem using the Pekeris branch cut.

[See Jensen, Fig 5.8.
Modes 1 and 4 are normal modes;
Modes 10 and 12 are virtual modes]

Pekeris waveguide Problem

$$\Psi(z) = A\sin(k_z z)\ ,$$

$$k_z = \sqrt{\left(\frac{\omega}{c}\right)^2 - k_r^2}\ .$$

Characteristic Equation

$$\tan(k_z D) = -\frac{i\rho_b\, k_z}{\rho\, k_{z,b}}\ ,$$

Modal Field Contribution

$$p = \left(e^{ik_{zm}z} + e^{-ik_{zm}z}\right) e^{ik_{rm}r}\ .$$